

# MONODROMY OF THE CASIMIR CONNECTION OF A SYMMETRISABLE KAC–MOODY ALGEBRA

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ABSTRACT. Let  $\mathfrak{g}$  be a symmetrisable Kac–Moody algebra and  $V$  an integrable  $\mathfrak{g}$ –module in category  $\mathcal{O}$ . We show that the monodromy of the (normally ordered) rational Casimir connection on  $V$  can be made equivariant with respect to the Weyl group  $W$  of  $\mathfrak{g}$ , and therefore defines an action of the braid group  $B_W$  of  $W$  on  $V$ . We then prove that this action is uniquely equivalent to the quantum Weyl group action of  $B_W$  on a quantum deformation of  $V$ , that is an integrable, category  $\mathcal{O}$ –module  $\mathcal{V}$  over the quantum group  $U_h\mathfrak{g}$  such that  $\mathcal{V}/h\mathcal{V}$  is isomorphic to  $V$ . This extends a result of the second author which is valid for  $\mathfrak{g}$  semisimple.

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## 1. INTRODUCTION

1.1. Let  $\mathfrak{g}$  be a complex, semisimple Lie algebra,  $(\cdot, \cdot)$  an invariant inner product on  $\mathfrak{g}$ ,  $\mathfrak{h} \subset \mathfrak{g}$  a Cartan subalgebra, and  $R \subset \mathfrak{h}^*$  the corresponding root system. Set  $\mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha \in R} \text{Ker}(\alpha)$ , and let  $V$  be a finite-dimensional representation of  $\mathfrak{g}$ . The Casimir connection of  $\mathfrak{g}$  is the flat connection on the holomorphically trivial vector bundle  $\mathbb{V}$  over  $\mathfrak{h}_{\text{reg}}$  with fibre  $V$  given by

$$\nabla_{\kappa} = d - \frac{\hbar}{2} \sum_{\alpha \in R_+} \frac{d\alpha}{\alpha} \cdot \mathcal{K}_{\alpha} \quad (1.1)$$

Here,  $\hbar$  is a complex deformation parameter,  $R_+ \subset R$  a chosen system of positive roots,<sup>1</sup> and  $\mathcal{K}_{\alpha} \in U\mathfrak{g}$  the truncated Casimir operator of the three-dimensional subalgebra  $\mathfrak{sl}_2^{\alpha} \subset \mathfrak{g}$  corresponding to  $\alpha$  given by

$$\mathcal{K}_{\alpha} = x_{\alpha}x_{-\alpha} + x_{-\alpha}x_{\alpha}$$

where  $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$  are root vectors such that  $(x_{\alpha}, x_{-\alpha}) = 1$  [18, 16, 4, 11]. Although the Weyl group  $W$  of  $\mathfrak{g}$  does not act on  $V$  in general, the action of the Tits extension of  $W$  on  $V$  can be used to twist the vector bundle  $(\mathbb{V}, \nabla_{\kappa})$  so that it becomes a  $W$ -equivariant, flat vector bundle  $(\tilde{\mathbb{V}}, \tilde{\nabla}_{\kappa})$ . Its monodromy gives rise to a one-parameter family of actions  $\mu_{\hbar}$  of the braid group  $B_W = \pi_1(\mathfrak{h}_{\text{reg}}/W)$  on  $V$  [18, 16].

1.2. A theorem of the second author asserts that the monodromy of  $\tilde{\nabla}_{\kappa}$  is described by the quantum group  $U_{\hbar}\mathfrak{g}$ , with deformation parameter given by  $\hbar = \pi i \hbar$  [19, 20]. Specifically, if  $\mathcal{V}$  is a quantum deformation of  $V$ , that is a  $U_{\hbar}\mathfrak{g}$ -module which is topologically free over  $\mathbb{C}[[\hbar]]$  and such that  $\mathcal{V}/\hbar\mathcal{V} \cong V$  as  $U\mathfrak{g}$ -modules, the action of  $B_W$  on  $V[[\hbar]]$  given by the formal Taylor series of  $\mu^{\hbar}$  at  $\hbar = 0$  is equivalent to that on  $\mathcal{V}$  given by the quantum Weyl group operators of  $U_{\hbar}\mathfrak{g}$ .

1.3. The goal of the present paper is to extend the description of the monodromy of the Casimir connection in terms of quantum groups to the case of an arbitrary symmetrisable Kac–Moody algebra  $\mathfrak{g}$ . This extension requires several new ideas, which we discuss below.

When the root system of  $\mathfrak{g}$  is infinite, the sum in (1.1) does not converge. This is easily overcome, however, by replacing each Casimir operator by its normally ordered version

$$:\mathcal{K}_{\alpha} := 2x_{-\alpha}^{(i)}x_{\alpha}^{(i)}$$

where  $\{x_{\pm\alpha}^{(i)}\}_{i=1}^{\dim \mathfrak{g}_{\pm\alpha}}$  are dual bases of the root spaces  $\mathfrak{g}_{\pm\alpha}$ , and  $\alpha$  is assumed positive. Although still infinite, the sum in

$$:\nabla_{\kappa} := d - \hbar \sum_{\alpha \in R_+} \frac{d\alpha}{\alpha} \cdot x_{-\alpha}^{(i)}x_{\alpha}^{(i)} \quad (1.2)$$

is now locally finite, provided the representation  $V$  lies in category  $\mathcal{O}$ . Moreover, the connection  $:\nabla_{\kappa}$  is flat [11] (we give an alternative proof of this, along the lines of its finite-dimensional counterpart, in Section 2).

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<sup>1</sup> $\nabla_{\kappa}$  is in fact independent of the choice of  $R_+$

1.4. Although it restores convergence, the normal ordering in (1.2) breaks the  $W$ -equivariance of the connection  $:\nabla_\kappa:$ . The lack of equivariance of its monodromy is measured by a 1-cocycle  $\mathcal{A} = \{\mathcal{A}_w\}$  on  $W$ , given by the monodromy of the abelian connection  $d - a_w$ , where

$$a_w = w_* : \nabla_\kappa : - : \nabla_\kappa : = -\hbar \sum_{\alpha \in \mathbb{R}_+ \cap w\mathbb{R}_-} \frac{d\alpha}{\alpha} \cdot \nu^{-1}(\alpha)$$

where  $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$  is the identification given by the bilinear form  $(\cdot, \cdot)$ . To rectify this, we prove in Section 2 that  $\mathcal{A}$  is the coboundary of an explicit abelian cochain  $\mathcal{B}$ . As a consequence, the monodromy of  $:\nabla_\kappa:$  multiplied by  $\mathcal{B}$  gives rise to a 1-parameter family of actions of  $B_W$  on any integrable, category  $\mathcal{O}$  module  $V$ .

When  $\mathfrak{g}$  is finite-dimensional,  $\mathcal{B}$  is (essentially) the monodromy of the abelian connection

$$d - \frac{\hbar}{2} \sum_{\alpha \in \mathbb{R}_+} \frac{d\alpha}{\alpha} \cdot (\mathcal{K}_\alpha - : \mathcal{K}_\alpha : ) = d - \frac{\hbar}{2} \sum_{\alpha \in \mathbb{R}_+} \frac{d\alpha}{\alpha} \nu^{-1}(\alpha)$$

In Appendix A, which may be of independent interest, we show that the latter expression can effectively be resummed when  $\mathfrak{g}$  is affine, thus giving an alternative construction of the cochain  $\mathcal{B}$  in this case. Our construction is based on the well-known resummation of the series  $\sum_{n \geq 0} (z + n)^{-1}$  via the logarithmic derivative  $\Psi$  of the Gamma function, through its expansion

$$\Psi(z) = \frac{1}{z} + \sum_{n \geq 1} \left( \frac{1}{z + n} - \frac{1}{n} \right)$$

1.5. The main result of this paper is that the ( $W$ -equivariant) monodromy of  $:\nabla_\kappa:$  is described by the quantum Weyl group operators acting on integrable, category  $\mathcal{O}$ -representations of the quantum group  $U_\hbar \mathfrak{g}$ . Our strategy is patterned on that of [19, 20], and hinges on the notion of braided quasi-Coxeter category. Such a category is, informally speaking, a braided tensor category which carries commuting actions of Artin's braid groups  $B_n$ , and of a given generalised braid group  $B_W$ , on the tensor powers of its objects. For  $U_\hbar \mathfrak{g}$ , this structure arises on the category  $\mathcal{O}_\hbar^{\text{int}}$  of integrable, highest weight modules via the  $R$ -matrices of all Levi subalgebras of  $U_\hbar \mathfrak{g}$ , and its quantum Weyl group operators. For the category  $\mathcal{O}^{\text{int}}$  of integrable, highest weight  $\mathfrak{g}$ -modules, we adapt the argument of [20] to show that such a structure arises from monodromy of the KZ equations of all Levi subalgebras of  $\mathfrak{g}$  and that of its Casimir connection. The description of the monodromy of the Casimir connection in terms of quantum Weyl group operators is then deduced by proving that  $\mathcal{O}_\hbar^{\text{int}}$  and  $\mathcal{O}^{\text{int}}$  are equivalent as quasi-Coxeter categories.

1.6. Such a statement naturally presupposes that  $\mathcal{O}_\hbar^{\text{int}}$  and  $\mathcal{O}^{\text{int}}$  are equivalent as abelian categories. When  $\mathfrak{g}$  is finite-dimensional, this follows from the fact that  $U_\hbar \mathfrak{g}[[\hbar]]$  and  $U_\hbar \mathfrak{g}$  are isomorphic as algebras. This is no longer true for a general  $\mathfrak{g}$ , but an equivalence of categories can be obtained via the Etingof–Kazhdan quantisation functor [7, 8], which realises  $U_\hbar \mathfrak{g}$  as (a subalgebra of) the endomorphisms of a suitable fiber functor on  $\mathcal{O}$ . The EK equivalence, however, is not compatible with the inclusion of Levi subalgebras, something required by the yoga of quasi-Coxeter categories. In [1], we modified this equivalence by constructing a relative version of the Etingof–Kazhdan functor which takes as input an inclusion of Lie bialgebras.

The main result of [1] is that the quasi-Coxeter structure on  $\mathcal{O}_h$  can be transferred to one on  $\mathcal{O}$ .

1.7. We proved in [2] that the corresponding quasi-Coxeter structure on  $\mathcal{O}$  is extremely rigid, namely that it is unique up to a unique twist. Thus, the stated equivalence follows once it is proved that the monodromy of the KZ equations for all Levi subalgebras of  $\mathfrak{g}$  and that of the Casimir equations arise from a braided quasi-Coxeter structure on  $\mathcal{O}$ . This is proved in [20] for a finite-dimensional Lie algebra  $\mathfrak{g}$ , but the proof carries over to an arbitrary  $\mathfrak{g}$ , provided one considers both the KZ and Casimir equations with values in a double holonomy algebra which contains the Lie algebra of the pure Artin braid groups and of the generalised pure braid group  $P_W$  corresponding to  $W$ . The proof is then completed by noticing that this holonomy algebra naturally maps to the universal PROPic algebra introduced in [2] and which contains the data underlying the quasi-Coxeter structure of the transported  $\mathcal{O}_h^{\text{int}}$ . This former fact is similar to the fact that the Lie algebra of the pure braid groups  $P_n$  map to the Hochschild complex of the universal enveloping algebra  $U\mathfrak{g}$ .

1.8. **Outline of the paper.** In Section 2, we review the definition of the (normally ordered) Casimir connection of a symmetrisable Kac-Moody algebra  $\mathfrak{g}$ , give an alternative proof of its flatness, and prove that its monodromy can be modified by an abelian cochain so as to define representations of the braid group  $B_W$  of the Weyl group  $W$  of  $\mathfrak{g}$ .

In Section 3, we show that the (appropriately modified) monodromy of the Casimir connection arises from a quasi-Coxeter structure on integrable, category  $\mathcal{O}$  representations of  $\mathfrak{g}$ . This follows by a straightforward adaptation of the De Concini-Procesi construction of fundamental solutions of holonomy equations to the case of an infinite hyperplane arrangement.

In Section 4, we review the definition of braided quasi-Coxeter category, and introduce a double holonomy algebra containing the coefficients of the universal version of both the KZ and Casimir connections.

In Section 5, we adapt the construction of the fusion operator of [20] so that it takes values in our double holonomy algebra, and show that it gives rise to a braided quasi-Coxeter structure on  $\mathcal{O}^{\text{int}}$  which gives rise to the monodromy of the KZ and Casimir equations.

In Section 6, we describe the transfer of braided quasi-Coxeter structure from  $\mathcal{O}_h^{\text{int}}$  to  $\mathcal{O}$ , following [1] and [2].

Section 7 contains our main result about the equivalence of braided quasi-Coxeter categories between  $\mathcal{O}_h^{\text{int}}$  and  $\mathcal{O}$ .

Appendix 6.1 reviews the basic definitions of PROPs, Lie bialgebra and Drinfeld-Yetter modules.

Finally, in Appendix A we show that, in the case of an affine Kac-Moody algebra  $\mathfrak{g}$ , the normally ordered Casimir connection can be modified by adding an explicit closed,  $\mathfrak{h}$ -valued one-form such that the sum of the two is equivariant under  $W$ .

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## 2. THE CASIMIR CONNECTION OF A SYMMETRISABLE KAC–MOODY ALGEBRA

**2.1. The Casimir connection.** Let  $\mathfrak{g}$  be a symmetrisable Kac–Moody algebra with Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$ ,  $R = \{\alpha\} \subset \mathfrak{h}^*$  the root system of  $\mathfrak{g}$  and set

$$\mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha \in R} \text{Ker}(\alpha)$$

Let  $R_+ \subset R$  be the set of positive roots of  $\mathfrak{g}$ . For any  $\alpha \in R_+$ , let  $\mathfrak{g}_{\pm\alpha} \subset \mathfrak{g}$  be the root subspaces corresponding to  $\pm\alpha$  and let  $\{e_{\pm\alpha}^{(i)}\}_{i=1}^{\dim \mathfrak{g}_\alpha}$  be bases of  $\mathfrak{g}_{\pm\alpha}$  which are dual to each other with respect to the invariant inner product  $(\cdot, \cdot)$  on  $\mathfrak{g}$ . Set

$$\mathcal{K}_\alpha^+ = 2 \sum_{i=1}^{\dim \mathfrak{g}_\alpha} e_{-\alpha}^{(i)} e_\alpha^{(i)}$$

Let  $V$  be a  $\mathfrak{g}$ -module lying in category  $\mathcal{O}$  and let  $\mathbb{V} = V \times \mathfrak{h}_{\text{reg}}$  be the holomorphically trivial vector bundle over  $\mathfrak{h}_{\text{reg}}$  with fibre  $V$ . Finally, let  $\mathfrak{h} \in \mathbb{C}$  be a complex parameter.

**Definition.** The Casimir connection of  $\mathfrak{g}$  is the connection on  $\mathbb{V}$  given by

$$\nabla_\kappa = d - \frac{\mathfrak{h}}{2} \sum_{\alpha \in R_+} \frac{d\alpha}{\alpha} \cdot \mathcal{K}_\alpha^+ \quad (2.1)$$

The Casimir connection for a semisimple Lie algebra was introduced, and shown to be flat by De Concini [4] and, independently by Millson–Toledano Laredo [18, 16] and Felder–Markov–Tarasov–Varchenko [11]. In [11], the case of an arbitrary symmetrisable Kac–Moody algebra was considered. We give an alternative proof of flatness in this more general case, along the lines of [18, 16] in Section 2.3.

**2.2. Local finiteness.** Note that the sum in (2.1) is locally finite even if  $R$  is infinite since, for any  $v \in V$ ,  $\mathcal{K}_\alpha^+ v = 0$  for all but finitely many  $\alpha \in R_+$ . Differently said, let  $\Pi = \{\alpha_i\} \subset R_+$  be the set of simple roots of  $\mathfrak{g}$ ,  $\deg : \Pi \rightarrow \mathbb{Z}_+$  a function such that  $\deg^{-1}(n)$  is finite for any  $n \in \mathbb{Z}_+$ , and extend  $\deg$  to a  $\mathbb{Z}_+$ -valued function on  $R_+$  by linearity. Let  $\lambda_1, \dots, \lambda_p \in \mathfrak{h}^*$  be such that the set of weights of  $V$  is contained in the finite union  $\bigcup_{i=1}^p D(\lambda_i)$  where  $D(\lambda_i) = \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda_i\}$ . Set, for any  $n \in \mathbb{Z}_+$ ,

$$V^n = \bigoplus_{\substack{\mu \in \mathfrak{h}^* : \\ \deg(\lambda_i - \mu) \leq n, \\ \forall i : \mu \in D(\lambda_i)}} V[\mu]$$

where  $V[\mu]$  is the weight space of  $V$  of weight  $\mu$ . Then,  $V = \varinjlim V^n$ , each  $V^n$  is invariant under the operators  $\mathcal{K}_\alpha^+$  and  $\mathcal{K}_\alpha^+$  acts as zero on  $V^n$  if  $\deg(\alpha) > n$ . Thus, if  $\mathbb{V}^n = V^n \times \mathfrak{h}_{\text{reg}}$  is the trivial vector bundle over  $\mathfrak{h}_{\text{reg}}$  with fibre  $V^n$ , then  $\mathbb{V} = \varinjlim \mathbb{V}^n$  and  $\nabla_\kappa = \varinjlim \nabla_\kappa^n$  where

$$\nabla_\kappa^n = d - \frac{\mathfrak{h}}{2} \sum_{\alpha \in R_+^n} \frac{d\alpha}{\alpha} \cdot \mathcal{K}_\alpha^+ \quad (2.2)$$

and  $R_+^n$  is the finite set

$$R_+^n = \{\alpha \in R_+ \mid \deg(\alpha) \leq n\} \quad (2.3)$$

Note also that the pair  $(\mathbb{V}^n, \nabla_\kappa^n)$  descends to a (trivial) vector bundle with connection on the complement  $\mathfrak{h}_{\text{reg}}^n$  of the hyperplanes  $\text{Ker}(\alpha)$ ,  $\alpha \in \mathbb{R}_+^n$ , in the finite-dimensional vector space

$$\mathfrak{h}^n = \mathfrak{h}/(\mathbb{R}_+^n)^\perp \quad (2.4)$$

Finally note that, due to the existence of proportional roots, the forms  $\frac{d\alpha}{\alpha}$  need not be pairwise distinct. For example, all positive imaginary roots  $m\delta$ ,  $m \in \mathbb{Z}_+$ , of an affine Kac–Moody algebra give rise to the same one-form  $d\delta/\delta$ .

### 2.3. Flatness.

**Theorem.** *The connection  $\nabla_\kappa$  is flat for any  $\mathfrak{h} \in \mathbb{C}$ .*

*Proof.* It suffices to prove that the connection  $\nabla_\kappa^n$  defined by (2.2) is flat for any  $n$ . Since  $\nabla_\kappa^n$  is pulled back from the finite-dimensional vector space  $\mathfrak{h}^n$  (2.4), Kohno’s lemma [13] implies that the flatness of  $\nabla_\kappa^n$  is equivalent to proving that, for any two-dimensional subspace  $U \subset \mathfrak{h}^*$  spanned by a subset of  $\mathbb{R}_+^n$ , the following holds on  $V^n$  for any  $\alpha \in U \cap \mathbb{R}_+^n$

$$[\mathcal{K}_\alpha^+, \sum_{\beta \in U \cap \mathbb{R}_+^n} \mathcal{K}_\beta^+] = 0$$

Since  $\mathcal{K}_\alpha^+$  acts as 0 on  $V^n$  if  $\deg(\beta) > n$ , this amounts to proving that, on  $V^n$

$$[\mathcal{K}_\alpha^+, \sum_{\beta \in U \cap \mathbb{R}_+} \mathcal{K}_\beta^+] = 0$$

Let

$$\mathfrak{g}_U = \mathfrak{h} \oplus \bigoplus_{\alpha \in U \cap \mathbb{R}} \mathfrak{g}_\alpha$$

be the subalgebra spanned by  $\mathfrak{h}$  and the root subspaces corresponding to the elements of  $U \cap \mathbb{R}$ . Then  $\mathfrak{g}_U$  is a generalized Kac–Moody algebra and, modulo terms in  $U\mathfrak{h}$ , the operator  $\sum_{\beta \in U \cap \mathbb{R}_+} \mathcal{K}_\beta^+$  is its Casimir operator. Since any element in  $U\mathfrak{h}$  commutes with  $\mathcal{K}_\alpha^+$ , the above commutator is therefore zero.  $\square$

**2.4. Fundamental group of root system arrangements.** Let  $\mathfrak{h}^e = \mathfrak{h}/\mathfrak{z}(\mathfrak{g})$  be the *essential* Cartan. Let  $\mathcal{C}_\mathbb{R}$  be the fundamental Weyl chamber in  $\mathfrak{h}_\mathbb{R}^e$ , and let

$$\mathbb{Y}_\mathbb{R} = \bigcup_{w \in W} w(\overline{\mathcal{C}_\mathbb{R}})$$

be the real Tits cone. Set  $\mathbb{Y} = \mathring{\mathbb{Y}}_\mathbb{R} + i\mathfrak{h}_\mathbb{R}^e$ . The Weyl group  $W$  acts properly discontinuously on  $\mathbb{Y}$  [14, 22]. The regular points of this action are the points of

$$\mathbb{X} = \mathbb{Y} \setminus \bigcup_{\alpha \in \mathbb{R}_+} \text{Ker}(\alpha)$$

The action on  $\mathbb{X}$  is now proper and free, and the space  $\mathbb{X}/W$  is a complex manifold. Let  $p : \mathbb{X} \rightarrow \mathbb{X}/W$  be the canonical projection. Fix a point  $x_0 \in \mathcal{IC}$ , and set  $x = p(x_0)$  as a base point in  $\mathbb{X}/W$ . For  $i = 1, \dots, n$ , define the curve  $\gamma_i : x_0 \rightarrow s_i(x_0)$  in  $\mathbb{X}$  by

$$\gamma_i(t) = \nu(t) + ((1-t)x_0 + s_i(x_0))$$

where  $\nu : [0, 1] \rightarrow \mathfrak{h}$ , with  $\nu(0) = 0 = \nu(1)$ . Finally, set  $S_i = p \circ \gamma_i \in \pi_1(\mathbb{X}/W, x)$ . The following result is due to van der Lek [21], and generalises Brieskorn’s Theorem [3] to the case of an arbitrary Weyl group.

**Theorem.** *The fundamental group of  $X/W$  is the Artin braid group  $B_W$ . More precisely,  $\pi_1(X/W)$  is presented on the generators  $S_1, \dots, S_l$ , with relations for any  $i, j \in I$  with  $m_{ij} < +\infty$  given by*

$$\underbrace{S_i S_j S_i \cdots}_{m_{ij}} = \underbrace{S_j S_i S_j \cdots}_{m_{ij}} \quad (2.5)$$

**2.5. Twisting of  $(\mathbb{V}, \nabla_\kappa)$ .** Let  $W \subset GL(\mathfrak{h})$  be the Weyl group of  $\mathfrak{g}$ . It is well known that  $W$  does not act on  $V$  in general, but that the triple exponentials

$$\tilde{s}_i = \exp(e_{\alpha_i}) \exp(-f_{\alpha_i}) \exp(e_{\alpha_i}) \quad (2.6)$$

corresponding to the simple roots  $\alpha_1, \dots, \alpha_r$  in  $R_+$  and a choice of  $\mathfrak{sl}_2$ -triples  $e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_i} \in \mathfrak{sl}_2^{\alpha_i}$  give rise to an action of an extension  $\widetilde{W}$  of  $W$  by the sign group  $\mathbb{Z}_2^r$  [17].

The flat vector bundle  $(\mathbb{V}, \nabla_\kappa)$  is equivariant under  $\widetilde{W}$ , and may be twisted into a  $W$ -equivariant, flat vector bundle  $(\widetilde{\mathbb{V}}, \widetilde{\nabla}_\kappa)$  on  $\mathfrak{h}_{\text{reg}}$  as follows [16]. Let  $\widetilde{\mathfrak{h}}_{\text{reg}} \xrightarrow{p} \mathfrak{h}_{\text{reg}}$  be the universal cover of  $\mathfrak{h}_{\text{reg}}$  and  $\mathfrak{h}_{\text{reg}}/W$ . Since the Tits extension  $\widetilde{W}$  is a quotient of the braid group  $B_W = \pi_1(\mathfrak{h}_{\text{reg}}/W)$ , the latter acts on the flat vector bundle  $p^*(\mathbb{V}, \nabla_\kappa)$  on  $\widetilde{\mathfrak{h}}_{\text{reg}}$ . By definition,  $(\widetilde{\mathbb{V}}, \widetilde{\nabla}_\kappa)$  is the quotient  $p^*(\mathbb{V}, \nabla_\kappa)/PB_W$ , where  $PB_W = \pi_1(\mathfrak{h}_{\text{reg}})$  is the pure braid group corresponding to  $W$ , and carries a residual action of  $W = B_W/PB_W$ .

**2.6. Monodromy representations.** The monodromy of  $(\widetilde{\mathbb{V}}, \widetilde{\nabla}_\kappa)$ , which we shall abusively refer to as the monodromy of the Casimir connection, yields a one-parameter family of representations  $\mu_V^h$  of the pure braid group  $PB_W$  on  $V$  which is obtained as follows. Fix a base point  $\tilde{x}_0 \in \widetilde{\mathfrak{h}}_{\text{reg}}$ , and let  $x_0, [x_0]$  be its images in  $\mathfrak{h}_{\text{reg}}$  and  $\mathfrak{h}_{\text{reg}}/W$  respectively. The braid group  $\pi_1(\mathfrak{h}_{\text{reg}}/W; [x_0])$  acts on fundamental solutions  $\Psi : \widetilde{\mathfrak{h}}_{\text{reg}} \rightarrow GL(V)$  of  $p^*\nabla_\kappa$  by  $b \bullet \Psi(\tilde{x}) = b \cdot \Psi(\tilde{x} \cdot b)$ . If  $\Psi$  is a given fundamental solution, then  $\mu_\Psi^h(b) = \Psi^{-1} \cdot b \bullet \Psi$  is locally constant function with values in  $GL(V)$  and the required monodromy.

**2.7. Equivariant extensions.** As pointed out, the Casimir connection  $\nabla_\kappa$  is not  $W$ -equivariant. We will prove below, however, that its monodromy can be corrected so as to become  $W$ -equivariant.

Let  $x_0$  be a fixed basepoint in the fundamental chamber  $\mathcal{C}$  in  $X$ . For each  $i \in I$ , let  $\gamma_i$  be a fixed elementary path from  $x_0$  to  $s_i(x_0)$  around the wall  $\alpha_i = 0$ . For any  $i \in I, w \in W$ , defined the path  $\gamma_{w,i}$  by

$$\gamma_{w,i} = w(\gamma_i) : w(x_0) \rightarrow (ws_i)(x_0)$$

Let  $P(X)$  be the groupoid with objects  $\{w(x_0)\}_{w \in W}$ . The morphisms are defined as follows. For any point  $w_1(x_0) \in P(X)$ , the morphisms  $P(X)(x_0, w_1(x_0))$  are given by the all paths from  $x_0$  to  $w_1(x_0)$  obtained by composition of the elements  $\gamma_{w,i}$ . We then define

$$P(X)(w_2(x_0), w_1(x_0)) = w_2 P(X)(x_0, w_2^{-1} w_1(x_0))$$

Finally, let  $H(X)$  be the groupoid with same objects of  $P(X)$  and morphisms

$$H(X)(x, y) = P(X)(x, y) / \sim$$

where  $\gamma \sim \gamma'$  if  $\gamma, \gamma'$  are homotopic in  $X$ .

**Theorem.** *The monodromy representation  $\mu$  extends to a representation of  $B_W \simeq \pi_1(X/W)$ . In particular, there exists a morphism  $\pi_1(X/W) \rightarrow GL(V)$  such that*

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{\mu} & GL(V) \\ \downarrow & \nearrow & \\ \pi_1(X/W) & & \end{array}$$

*is commutative.*

*Proof.* The monodromy of the Casimir connection  $\nabla$  provides a homomorphism

$$\mu : H(X) \rightarrow GL(V)$$

Let  $\mu_w$  denote the monodromy representation of the connection  $w^*\nabla$ . Let

$$\alpha_w : H(X) \rightarrow GL(V)$$

be the map  $\alpha_w = \mu\mu_w^{-1}$ , which measures the lack of  $W$ -equivariance of  $\nabla$  and describes the monodromy of the abelian connection

$$\nabla_w^h = d - \sum_{\substack{\alpha > 0 \\ w\alpha < 0}} t_\alpha d \log(\alpha)$$

where  $t_\alpha = \nu^{-1}(\alpha)$ . In order to define a morphism  $\pi_1(X/W) \rightarrow GL(V)$  satisfying the required property, it is enough to construct a morphism  $\beta : H(X) \rightarrow GL(V)$  such that

$$\alpha_w = \beta_w \beta^{-1} \quad (2.7)$$

where  $\beta_w(\gamma) = w^{-1}(\beta(w(\gamma)))$ .

The choice of elements  $\{\beta(\gamma_i)\}_{i \in I}$  determines automatically a morphism  $\beta : P(X) \rightarrow GL(V)$ . The result follows by

**Lemma.** *There exists a choice of the elements  $\{\beta(\gamma_i)\}_{i \in I}$  such that  $\beta : P(X) \rightarrow GL(V)$  preserves the homotopy relations in  $P(X)$  and descends to a morphism  $\beta : H(X) \rightarrow GL(V)$ . In particular, the choice (for any  $a, b \in \mathbb{C}$ )*

$$\beta(\gamma_i) = e^{ah_i + bh_i^2} \quad (2.8)$$

*is allowed.*

□

**2.7.1. Proof of Lemma 2.7.** For any reduced expression  $\underline{s} = (i_1, \dots, i_n)$  of  $w \in W$ , there is a canonical element  $\gamma_{\underline{s}} \in P(X)$  from  $x_0$  to  $w(x_0)$

$$\gamma_{\underline{s}} = \prod_{k=1, \dots, n}^{\leftarrow} s_{i_1} \cdots s_{i_{k-1}}(\gamma_{i_k})$$

and

$$\begin{aligned} \beta(\gamma_{\underline{s}}) &= \prod_{k=1}^n \beta(s_{i_1} \cdots s_{i_{k-1}}(\gamma_{i_k})) = \\ &= \left( \prod_{k=1}^n s_{i_1} \cdots s_{i_{k-1}}(\alpha_{s_{i_1} \cdots s_{i_{k-1}}}(\gamma_{i_k})) \right) \left( \prod_{k=1}^n s_{i_1} \cdots s_{i_{k-1}}(\beta(\gamma_{i_k})) \right) \end{aligned}$$



We have to show that the map  $\beta$  defined by the choice (2.8) preserves the homotopy relations in  $P(X)$ . Following [21], it is enough to prove that  $\beta$  is independent of the reduced expression  $\underline{s}$ .

It is easy to see that, if we choose  $\beta(\gamma_i)$  as in (2.8),

$$\prod_{k=1}^n s_{i_1} \cdots s_{i_{k-1}}(\beta(\gamma_{i_k})) = \prod_{\substack{\alpha > 0 \\ w^{-1}\alpha < 0}} e^{ah_\alpha + bh_\alpha^2}$$

and it is therefore independent of the reduced expression of  $w$ . Set  $w_{k-1} = s_{i_1} \cdots s_{i_{k-1}}$ . Then

$$w_{k-1}(\alpha_{w_{k-1}(\gamma_{i_k}))} = \prod_{\substack{\alpha > 0 \\ w_{k-1}(\alpha) < 0}} \alpha(x_0)^{-t_{w_{k-1}(\alpha)}} \cdot s_{i_k}(\alpha)(x_0)^{t_{w_{k-1}(\alpha)}}$$

Set  $I_k = \{\alpha > 0 \mid w_k(\alpha) < 0\}$ . It follows that <sup>1</sup>

$$\begin{aligned} \prod_{k=1}^n w_{k-1}(\alpha_{w_{k-1}(\gamma_{i_k}))} &= \prod_{k=1}^{n-1} \alpha_{i_k}(x_0)^{t_{-w_k(\alpha_{i_k})}} \cdot \prod_{\alpha \in I_{n-1}} s_{i_n}(\alpha)(x_0)^{-t_{-w_{n-1}(\alpha)}} = \\ &= \prod_{k=1}^n \alpha_{i_k}(x_0)^{t_{-w_k(\alpha_{i_k})}} \cdot \prod_{\substack{\alpha > 0 \\ w\alpha < 0}} \alpha(x_0)^{-t_{-w(\alpha)}} \end{aligned}$$

Therefore it remains to show that

$$A_{\underline{s}} = \prod_{k=1}^n \alpha_{i_k}(x_0)^{t_{-w_k(\alpha_{i_k})}}$$

is independent of the choice of the reduced expression  $\underline{s}$ .

Let  $\mathbf{R}^w$  be the diagram with vertices the reduced expressions of  $w$ . Two reduced expressions,  $\underline{s} = (i_1, \dots, i_n)$  and  $\underline{s}' = (j_1, \dots, j_n)$ , are connected by an edge if  $\underline{s}$  can be obtained by  $\underline{s}'$  by replacing a sequence of  $m$  indices  $i, j, i, \dots$  with  $m$  indices  $j, i, j, \dots$ , where  $m = m_{ij} < \infty$ . The diagram  $\mathbf{R}^w$  is connected and it is enough to show that  $A_{\underline{s}} = A_{\underline{s}'}$  whenever  $(\underline{s}, \underline{s}')$  is an edge. To see this, we are reduced to consider only the four cases of rank 2, with  $w = w_{ij}$ , *i.e.*, the longest elements of the subgroups  $W_{ij}$  generated by the simple reflections  $s_i, s_j$ .

(i) For  $A_1 \times A_1$ ,  $\underline{s} = (1, 2)$ ,  $\underline{s}' = (2, 1)$ , and<sup>2</sup>

$$A_{\underline{s}} = \alpha_1^{t_{\alpha_1}} \alpha_2^{t_{\alpha_2}} = \alpha_2^{t_{\alpha_2}} \alpha_1^{t_{\alpha_1}} = A_{\underline{s}'}$$

(ii) For  $A_2$ ,  $\underline{s} = (1, 2, 1)$ ,  $\underline{s}' = (2, 1, 2)$  and

$$A_{\underline{s}} = \alpha_1^{t_{\alpha_1}} \alpha_2^{t_{\alpha_1+\alpha_2}} \alpha_1^{t_{\alpha_2}} = \alpha_2^{t_{\alpha_2}} \alpha_1^{t_{\alpha_1+\alpha_2}} \alpha_2^{t_{\alpha_1}} = A_{\underline{s}'}$$

(iii) For  $B_2$ ,  $\underline{s} = (1, 2, 1, 2)$ ,  $\underline{s}' = (2, 1, 2, 1)$  and

$$A_{\underline{s}} = \alpha_1^{t_{\alpha_1}} \alpha_2^{t_{2\alpha_1+\alpha_2}} \alpha_1^{t_{\alpha_1+\alpha_2}} \alpha_2^{t_{\alpha_2}} = \alpha_2^{t_{\alpha_2}} \alpha_1^{t_{\alpha_1+\alpha_2}} \alpha_2^{t_{2\alpha_1+\alpha_2}} \alpha_1^{t_{\alpha_1}} = A_{\underline{s}'}$$

(iv) For  $G_2$ ,  $\underline{s} = (1, 2, 1, 2, 1, 2)$ ,  $\underline{s}' = (2, 1, 2, 1, 2, 1)$  and

$$\begin{aligned} A_{\underline{s}} &= \alpha_1^{t_{\alpha_1}} \alpha_2^{t_{3\alpha_1+\alpha_2}} \alpha_1^{t_{2\alpha_1+\alpha_2}} \alpha_2^{t_{3\alpha_1+2\alpha_2}} \alpha_1^{t_{\alpha_1+\alpha_2}} \alpha_2^{t_{\alpha_2}} = \\ &= \alpha_2^{t_{\alpha_2}} \alpha_1^{t_{\alpha_1+\alpha_2}} \alpha_2^{t_{3\alpha_1+2\alpha_2}} \alpha_1^{t_{2\alpha_1+\alpha_2}} \alpha_2^{t_{3\alpha_1+\alpha_2}} \alpha_1^{t_{\alpha_1}} = A_{\underline{s}'} \end{aligned}$$

<sup>1</sup>It is enough to observe that  $I_k \setminus s_{i_k} I_{k-1} = \{\alpha_{i_k}\}$ .

<sup>2</sup>in this paragraph only,  $\alpha$  stands for  $\alpha(x_0)$  for simplicity.

### 3. A DIFFERENTIAL QUASI-COXETER STRUCTURE ON CATEGORY $\mathcal{O}$

In this section, we review the definition of quasi-Coxeter category following [1]. We then prove that the monodromy of the Casimir connection defines such a structure on the category  $\mathcal{O}^{\text{int}}$  of integrable, highest weight  $\mathfrak{g}$ -modules. This structure is universal, in that it is constructed on the holonomy algebra  $\mathfrak{t}_{\mathbb{R}}$  of the root arrangement in  $\mathfrak{h}$ , and then transferred to  $\mathcal{O}$  via a morphism  $\mathfrak{t}_{\mathbb{R}} \rightarrow \text{End}(\mathfrak{f})$ , where  $\mathfrak{f}$  is the forgetful functor to vector spaces. The differential quasi-Coxeter structure on  $\mathfrak{t}_{\mathbb{R}}$  is obtained by adapting the De Concini-Procesi construction of the fundamental solution of the holonomy equations [5] to the case of an infinite hyperplane arrangement.

#### 3.1. Quasi-Coxeter categories.

**3.1.1. Diagrams and nested sets.** The terminology in 3.1.1–3.1.2 is taken from [19, Part I] and [1, Sec. 2], to which we refer for more details.

A *diagram* is a nonempty undirected graph  $D$  with no multiple edges or loops. We denote the set of vertices of  $D$  by  $\mathbb{V}(D)$ . A *subdiagram*  $B \subseteq D$  is a full subgraph of  $D$ , that is, a graph consisting of a (possibly empty) subset of vertices of  $D$ , together with all edges of  $D$  joining any two elements of it. Two subdiagrams  $B_1, B_2 \subseteq D$  are *orthogonal* if they have no vertices in common and no two vertices  $i \in B_1$ ,  $j \in B_2$  are joined by an edge in  $D$ .  $B_1$  and  $B_2$  are *compatible* if either one contains the other or they are orthogonal.

A *nested set* on a diagram  $D$  is a collection  $\mathcal{H}$  of pairwise compatible, connected subdiagrams of  $D$  which contains the connected components  $D_1, \dots, D_r$  of  $D$ .

Let  $\mathcal{N}_D$  be the partially ordered set of nested sets on  $D$ , ordered by reverse inclusion.  $\mathcal{N}_D$  has a unique maximal element  $\mathbf{1} = \{D_i\}_{i=1}^r$  and its minimal elements are the maximal nested sets. We denote the set of maximal nested sets on  $D$  by  $\text{Mns}(D)$ . Every nested set  $\mathcal{H}$  on  $D$  is uniquely determined by a collection  $\{\mathcal{H}_i\}_{i=1}^r$  of nested sets on the connected components of  $D$ . We therefore obtain canonical identifications

$$\mathcal{N}_D = \prod_{i=1}^r \mathcal{N}_{D_i} \quad \text{and} \quad \text{Mns}(D) = \prod_{i=1}^r \text{Mns}(D_i)$$

**3.1.2. Quotient diagrams.** Let  $B \subsetneq D$  be a proper subdiagram with connected components  $B_1, \dots, B_m$ .

**Definition.** The set of vertices of the *quotient diagram*  $D/B$  is  $\mathbb{V}(D) \setminus \mathbb{V}(B)$ . Two vertices  $i \neq j$  of  $D/B$  are linked by an edge if and only if the following holds in  $D$

$$i \not\leq j \quad \text{or} \quad i, j \not\leq B_i \quad \text{for some } i = 1, \dots, m$$

For any connected subdiagram  $C \subseteq D$  not contained in  $B$ , we denote by  $\overline{C} \subseteq D/B$  the connected subdiagram with vertex set  $\mathbb{V}(C) \setminus \mathbb{V}(B)$ .

For any  $B \subset B'$ , we denote by  $\text{Mns}(B', B)$  the collection of maximal nested sets on  $B'/B$ . For any  $B \subset B' \subset B''$ , there is an embedding

$$\cup : \text{Mns}(B'', B') \times \text{Mns}(B', B) \rightarrow \text{Mns}(B'', B)$$

such that, for any  $\mathcal{F} \in \text{Mns}(B'', B')$ ,  $\mathcal{G} \in \text{Mns}(B', B)$ ,

$$(\mathcal{F} \cup \mathcal{G})_{B'/B} = \mathcal{G}$$

3.1.3. *D-categories.* Recall [19, Section 3] that, given a diagram  $D$ , a  $D$ -algebra is a pair  $(A, \{A_B\}_{B \subset D})$ , where  $A$  is an associative algebra and  $\{A_B\}_{B \subset D}$  is a collection of subalgebras indexed by subdiagrams of  $D$ , and satisfying

$$A_B \subseteq A_{B'} \quad \text{if } B \subseteq B' \quad \text{and} \quad [A_B, A_{B'}] = 0 \quad \text{if } B \perp B'$$

The following rephrases the notion of  $D$ -algebras in terms of their category of representations.

**Definition.** A  $D$ -category

$$\mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\})$$

is the datum of

- a collection of  $k$ -linear additive categories  $\{\mathcal{C}_B\}_{B \subset D}$
- for any pair of subdiagrams  $B \subseteq B'$ , an additive  $k$ -linear functor  $F_{BB'} : \mathcal{C}_{B'} \rightarrow \mathcal{C}_B^1$
- for any  $B \subset B'$ ,  $B' \perp B''$ ,  $B', B'' \subset B'''$ , a homomorphism of  $k$ -algebras

$$\eta : \text{End}(F_{BB'}) \rightarrow \text{End}(F_{(B \cup B'')B'''})$$

satisfying the following properties

- For any  $B \subseteq D$ ,  $F_{BB} = \text{id}_{\mathcal{C}_B}$ .
- For any  $B \subseteq B' \subseteq B''$ ,  $F_{BB'} \circ F_{B'B''} = F_{BB''}$ .
- For any  $B \subset B'$ ,  $B' \perp B''$ ,  $B', B'' \subset B'''$ , the following diagram of algebra homomorphisms commutes:

$$\begin{array}{ccc} & \text{End}(F_{BB'}) & \\ \text{id}_{F_{B(B \cup B'')}} \otimes \eta \swarrow & & \searrow \text{id} \otimes \text{id}_{F_{B'B''}} \\ \text{End}(F_{BB'}) \otimes \text{End}(F_{B'B''}) & & \text{End}(F_{B(B \cup B'')}) \otimes \text{End}(F_{(B \cup B'')B'''}) \\ & \searrow \circ & \swarrow \circ \\ & \text{End}(F_{BB'''}) & \end{array}$$

**Remark.** It may seem more natural to replace the equality of functors  $F_{BB'} \circ F_{B'B''} = F_{BB''}$  by the existence of invertible natural transformations  $\alpha_{BB''}^{B'} : F_{BB'} \circ F_{B'B''} \Rightarrow F_{BB''}$  for any  $B \subseteq B'$  satisfying the associativity constraints  $\alpha_{BB'''}^{B'} \circ F_{BB'}(\alpha_{B'B'''}^{B''}) = \alpha_{BB'''}^{B''} \circ (\alpha_{BB''}^{B'})_{F_{B''B'''}}$  for any  $B \subseteq B' \subseteq B'' \subseteq B'''$ . A simple coherence argument shows however that this leads to a notion of  $D$ -category which is equivalent to the one given above.

**Remark.** We will usually think of  $\mathcal{C}_\emptyset$  as a base category and at the functors  $F$  as forgetful functors. Then the family of algebras  $\text{End}(F_B)$  defines, through the morphisms  $\alpha$ , a structure of  $D$ -algebra on  $\text{End}(F_D)$ . Conversely, every  $D$ -algebra  $A$  admits such a description setting  $\mathcal{C}_B = \text{Rep } A_B$  for  $B \neq \emptyset$  and  $\mathcal{C}_\emptyset = \text{Vect}_k$ ,  $F_{BB'} = i_{B'B}^*$ , where  $i_{B'B} : A_B \subset A_{B'}$  is the inclusion.

**Remark.** The conditions satisfied by the maps  $\eta$  imply that, given  $B = \bigsqcup_{j=1}^r B_j$ , with  $B_j \subset D$  connected and pairwise orthogonal, the images in  $\text{End}(F_B)$  of the maps

$$\text{End}(F_{B_j}) \rightarrow \text{End}(F_{B_j} F_{B_j B}) = \text{End}(F_B)$$

pairwise commute. This condition rephrases for the endomorphism algebras the  $D$ -algebra axiom

$$[A_{B'}, A_{B''}] = 0 \quad \forall \quad B' \perp B''$$

<sup>1</sup>When  $B = \emptyset$  we will omit the index  $B$ .

that is equivalent to the condition, for any  $B \supset B', B''$ ,

$$A_{B'} \subset A_B^{B''}$$

#### 3.1.4. Labelled diagrams and Artin braid groups.

**Definition.** A *labelling* of the diagram  $D$  is the assignment of an integer  $m_{ij} \in \{2, 3, \dots, \infty\}$  to any pair  $i, j$  of distinct vertices of  $D$  such that

$$m_{ij} = m_{ji} \quad m_{ij} = 2$$

if and only if  $i \perp j$ .

Let  $D$  be a labeled diagram.

**Definition.** The *Artin braid group*  $B_D$  is the group generated by elements  $S_i$  labeled by the vertices  $i \in D$  with relations

$$\underbrace{S_i S_j \cdots}_{m_{ij}} = \underbrace{S_j S_i \cdots}_{m_{ij}}$$

for any  $i \neq j$  such that  $m_{ij} < \infty$ . We shall also refer to  $B_D$  as the braid group corresponding to  $D$ .

#### 3.1.5. Quasi-Coxeter categories.

**Definition.** A *quasi-Coxeter category of type  $D$*

$$\mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\}, \{\Upsilon_{\mathcal{F}\mathcal{G}}\}, \{S_i\})$$

is the datum of

- a  $D$ -category  $\mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\})$
- for any elementary pair  $(\mathcal{F}, \mathcal{G})$  in  $\text{Mns}(B, B')$ , a natural transformation

$$\Upsilon_{\mathcal{F}\mathcal{G}} \in \text{Aut}(F_{BB'})$$

- for any vertex  $i \in V(D)$ , an element

$$S_i \in \text{Aut}(F_i)$$

satisfying the following conditions

- **Orientation.** For any  $\mathcal{F}, \mathcal{G}$ ,

$$\Upsilon_{\mathcal{F}\mathcal{G}} = \Upsilon_{\mathcal{G}\mathcal{F}}^{-1}$$

- **Transitivity.** For any pair  $\mathcal{F}, \mathcal{G}, \mathcal{H}$ .

$$\Upsilon_{\mathcal{F}\mathcal{G}} \Upsilon_{\mathcal{G}\mathcal{H}} = \Upsilon_{\mathcal{F}\mathcal{H}}$$

- **Factorization.** The assignment

$$\Upsilon : \text{Mns}(B, B')^2 \rightarrow \text{Aut}(F_{B'B})$$

is compatible with the embedding

$$\cup : \text{Mns}(B, B') \times \text{Mns}(B', B'') \rightarrow \text{Mns}(B, B'')$$

for any  $B'' \subset B' \subset B$ , i.e., the diagram

$$\begin{array}{ccc} \text{Mns}(B, B')^2 \times \text{Mns}(B', B'')^2 & \xrightarrow{\Upsilon \times \Upsilon} & \text{Aut}(F_{B''B'}) \times \text{Aut}(F_{B'B}) \\ \cup \downarrow & & \downarrow \circ \\ \text{Mns}(B, B'')^2 & \xrightarrow{\Upsilon} & \text{Aut}(F_{B''B}) \end{array}$$

is commutative.

- **Braid relations.** For any pairs  $i, j$  of distinct vertices of  $B$ , such that  $2 < m_{ij} < \infty$ , and elementary pair  $(\mathcal{F}, \mathcal{G})$  in  $\text{Mns}(B)$  such that  $i \in \mathcal{F}, j \in \mathcal{G}$ , the following relations hold in  $\text{End}(F_B)$

$$\text{Ad}(\Upsilon_{\mathcal{G}\mathcal{F}})(S_i) \cdot S_j \cdots = S_j \cdot \text{Ad}(\Upsilon_{\mathcal{G}\mathcal{F}})(S_i) \cdots$$

where, by abuse of notation, we denote by  $S_i$  its image in  $\text{End}(F_B)$  and the number of factors in each side equals  $m_{ij}$ .

### 3.2. Extended Kac–Moody algebras.

3.2.1. *Motivation.* A quasi–Coxeter category has an underlying structure of  $D$ –category, which generalise the notion of  $D$ –algebra. A first example is given by semisimple Lie algebras.

Let  $\mathfrak{g}$  be a semisimple Lie algebra with Dynkin diagram  $D$  and Chevalley generators  $\{e, f_i, h_i\}_{i \in D}$ . The collection of diagrammatic subalgebras  $\{\mathfrak{g}_B\}_{B \subset D}$ , where  $\mathfrak{g}_B$  is generated by  $\{e_i, f_i, h_i\}_{i \in B}$  defines a  $D$ –algebra structure on  $\mathfrak{g}$ .

This is not always true for symmetrisable Kac–Moody algebra. Indeed, it is easy to show that the Kac–Moody algebra corresponding to the symmetric irreducible Cartan matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

does not admit any  $D_{\mathfrak{g}}$ –algebra structure on  $\mathfrak{g}(A)$  [1].

3.2.2. *Extended Kac–Moody algebras.* Following the suggestion of P. Etingof, we give a modified definition of  $\mathfrak{g}$ , along the lines of [10], characterized by a bigger Cartan subalgebra.

Let  $A = (a_{ij})_{i,j \in I} \in M_n(\mathbb{C})$  be a symmetrisable generalised Cartan matrix.

**Definition.** The *extended Kac–Moody algebra* of  $A$  is the  $k$ –algebra  $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}(A)$  with generators  $e_i, f_i, h_i, \lambda_i^\vee$ ,  $i \in I$ , and defining relations

- $[h_i, h_j] = [\lambda_i^\vee, \lambda_j^\vee] = [h_i, \lambda_j^\vee] = 0$
- $[h_i, e_j] = a_{ij}e_j, [h_i, f_j] = -a_{ji}f_j$
- $[\lambda_i^\vee, e_j] = \delta_{ij}e_j, [\lambda_i^\vee, f_j] = -\delta_{ij}f_j$
- $\text{ad}(x_i)^{1-a_{ij}}(x_j) = 0$  for  $i \neq j$  and  $x = e, f$

**Proposition.** Let  $A$  be a symmetrisable generalized Cartan matrix of rank  $l$ ,  $D$  its Dynkin diagram,  $\mathfrak{g}$  the corresponding Kac–Moody algebra. Let  $\{d_r\}_{r=l+1}^n$  be the completion of  $\{h_i\}_{i=1}^n$  to a basis of  $\mathfrak{h} \subset \mathfrak{g}$  defined by the relations

$$[d_r, d_s] = 0 = [d_r, h_i] \quad [d_r, e_i] = \delta_{ir}e_i \quad [d_r, f_i] = -\delta_{ir}f_i$$

- (i)  $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}_D$  has a canonical structure of  $D$ –algebra, given by the collection of subalgebras  $\{\overline{\mathfrak{g}}_B\}_{B \subset D}$ , where

$$\overline{\mathfrak{g}}_B = \langle e_i, f_i, h_i, \lambda_i^\vee \mid i \in B \rangle$$

- (ii) There is a canonical embedding  $\mathfrak{g} \subset \overline{\mathfrak{g}}$  mapping

$$e_i, f_i, h_i, d_r \mapsto e_i, f_i, h_i, \lambda_r^\vee$$

$i = 1, \dots, n, r = l, \dots, n$ . At the level of Cartan subalgebras the inclusion is compatible with the symmetric, invariant, non-degenerate bilinear forms on  $\mathfrak{h}, \bar{\mathfrak{h}}$ .

Henceforth, we omit the adjective *extended*.

**3.3. The holonomy algebra.** Let  $\mathfrak{g}$  be a symmetrisable Kac–Moody algebra with root system  $R$  and Dynkin diagram  $D$ . Let  $F_R$  be the free algebra generated by the symbols  $t_\alpha, \alpha \in R_+$ . For any  $k \in \mathbb{Z}_{\geq 0}$  set

$$B_k = \{\alpha \mid \text{ht}(\alpha) \leq k\}$$

and let  $J_k$  be the two sided ideal in  $F_R$  generated by the symbols  $t_\alpha, \alpha \notin B_k$ . Set

$$\bar{F}_R = \lim_k F_R / J_k$$

**Definition.** The holonomy algebra  $t_R$  is the quotient of  $\bar{F}_R$  by the relations

$$[t_\alpha, \sum_{\beta \in \Psi} t_\beta] = 0$$

where  $\Psi \subset R$  is any rank two subsystem.

Let  $\tilde{J}_k$  be the two sided ideal generated by  $J_k$  and

$$[t_\alpha, \sum_{\beta \in \Psi \cap B_k} t_\beta]$$

where  $\Psi$  is as before.

**Lemma.** *There is a canonical isomorphism of algebras*

$$t_R \simeq \lim_k F_R / \tilde{J}_k$$

*Proof.* The canonical projections  $t_R \rightarrow F_R / \tilde{J}_k$  induce a surjective homomorphism

$$t_R \rightarrow \lim_k F / \tilde{J}_k$$

with trivial kernel. □

**3.3.1. Holonomy algebra and category  $\mathcal{O}$ .** Let  $\mathcal{O}$  be the category of deformation highest weight integrable  $\mathfrak{g}$ -modules,  $\mathcal{A}$  the category of topologically free  $\mathbb{C}[[\hbar]]$ -modules, and  $f : \mathcal{O} \rightarrow \mathcal{A}$  the forgetful functor.

**Proposition.** *The linear map  $\xi_R : t_R \rightarrow \mathcal{U}$  defined by*

$$\xi_R(t_\alpha) = \frac{\hbar}{2} \mathcal{K}_\alpha^+$$

*where  $\mathcal{K}_\alpha^+$  is the Wick-ordered (truncated) Casimir operator of  $\mathfrak{sl}_2^\alpha$ , is a morphism of algebras, compatible with the natural gradation on  $t_R$ .*

*Proof.* It follows from the commutation relations proved in 2.3. □

3.3.2. *Weak quasi-Coxeter structures on  $\mathfrak{t}_R$ .* Hereafter, unless otherwise stated, we will adopt the same notation  $\mathfrak{t}_R$  to denote the double holonomy algebra and its completion with respect to the grading.

For any subdiagram  $B \subset D$ , we denote by  $R_B$  the corresponding root subsystem, by  $\mathfrak{t}_B$  the holonomy algebra  $\mathfrak{t}_{R_B}$ . For any pair of subdiagrams  $B \subset B' \subset D$ , we denote by  $\mathfrak{t}_{B'}^B$  the subalgebra of  $\mathfrak{t}_B$ -invariant elements in  $\mathfrak{t}_{B'}$ .

**Definition.** A *weak quasi-Coxeter structure* on  $\mathfrak{t}_R$  is a collection of elements  $\Upsilon_{\mathcal{F}\mathcal{G}} \in \mathfrak{t}_R$ , referred to as *De Concini-Procesi associators*, for any  $B' \subseteq B$  and for any pair of maximal nested sets  $\mathcal{F}, \mathcal{G} \in \text{Mns}(B, B')$ , such that

$$\Upsilon_{\mathcal{F}\mathcal{G}} = 1 \pmod{(\mathfrak{t}_B)_{\geq 1}}$$

and satisfying the following properties

- **Orientation:** for any  $\mathcal{F}, \mathcal{G} \in \text{Mns}(B, B')$

$$\Upsilon_{\mathcal{F}\mathcal{G}} = \Upsilon_{\mathcal{G}\mathcal{F}}^{-1}$$

- **Transitivity:** for any  $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \text{Mns}(B, B')$

$$\Upsilon_{\mathcal{H}\mathcal{F}} = \Upsilon_{\mathcal{H}\mathcal{G}} \Upsilon_{\mathcal{G}\mathcal{F}}$$

- **Factorisation:**

$$\Upsilon_{(\mathcal{F}_1 \cup \mathcal{F}_2)(\mathcal{G}_1 \cup \mathcal{G}_2)} = \Upsilon_{\mathcal{F}_1 \mathcal{G}_1} \Upsilon_{\mathcal{F}_2 \mathcal{G}_2}$$

for any  $B'' \subseteq B' \subseteq B$ ,  $\mathcal{F}_1, \mathcal{G}_1 \in \text{Mns}(B, B')$  and  $\mathcal{F}_2, \mathcal{G}_2 \in \text{Mns}(B', B'')$ .

### 3.4. Monodromy of the Casimir connection.

3.4.1. *The Casimir connection.* Let

$$\mathfrak{c} = \{h \in \mathfrak{h} \mid (\alpha_i, h) = 0 \ \forall i \in D\}$$

be the center of  $\mathfrak{g}$  and set

$$\mathfrak{h}_{\text{reg}} = \mathfrak{h}/\mathfrak{c} \setminus \bigcup_{\alpha \in R_+} \ker(\alpha)$$

**Definition.** The Casimir connection is the flat connection on  $\mathfrak{h}_{\text{reg}}$  with values in  $\mathfrak{t}_R$

$$\nabla = d - \sum_{\alpha \in R_+} \frac{d\alpha}{\alpha} \mathfrak{t}_\alpha$$

The flatness of  $\nabla$  is proved as in 2.3.

3.4.2. *Holomorphic functions.* In order to perform the necessary analysis we consider the algebra  $\widehat{\mathfrak{t}}_R$ , completion of  $\mathfrak{t}_R$  with respect to the natural gradation  $\deg(\mathfrak{t}_\alpha) = 1$ . A solution of the *holonomy equation*

$$dG = \sum_{\alpha \in R_+} \frac{d\alpha}{\alpha} \mathfrak{t}_\alpha G \tag{3.1}$$

is a holomorphic functions (in its domain of definition) with values in  $\widehat{\mathfrak{t}}_R$ .

The analytic computations performed with functions with values in  $\widehat{\mathfrak{t}}_R$  are justified by the fact that  $\widehat{\mathfrak{t}}_R$  is the inverse limit of the finite dimensional algebras  $F_R/J_{k,n}$  where  $J_{k,n}$  is the ideal of the elements of degree  $\geq n$  in  $F_R/\widetilde{J}_k$ .

So a function is determined by a sequence of compatible functions in the finite dimensional algebras  $F_R/J_{k,n}$ .

**3.4.3. Blow-up coordinates.** Let  $\mathcal{F} \in \text{Mns}(D)$  be a maximal nested set on  $D$ . For any  $\alpha \in R$ , let  $p_{\mathcal{F}}(\alpha)$  be the minimal element  $B \in \mathcal{F}$  such that  $\text{supp}(\alpha) \subseteq B$ . Then  $p_{\mathcal{F}}$  establishes a one to one correspondence between the simple roots  $\{\alpha_1, \dots, \alpha_n\}$  and the subdiagrams in  $\mathcal{F}$ . For any  $B \in \mathcal{F}$ , we denote by  $\alpha_B$  the simple root corresponding to  $B$  under  $p$ . For any  $B \in \mathcal{F}$ , we denote by  $c_{\mathcal{F}}(B)$  the minimal element in  $\mathcal{F}$  which contains properly  $B$ . When no confusion is possible, we will avoid the index  $\mathcal{F}$  and we will write  $p(\alpha)$  and  $c(B)$  in place of  $p_{\mathcal{F}}(\alpha)$ ,  $c_{\mathcal{F}}(B)$ .

For any  $B \in \mathcal{F}$  set

$$x_B = \sum_{i \in B} \alpha_i$$

Then  $\{x_B\}_{B \in \mathcal{F}}$  defines a set of coordinates on  $\mathfrak{h}^e$ . Let  $U = \mathbb{C}^{\mathcal{F}}$  with coordinates  $\{u_B\}_{B \in \mathcal{F}}$ . Let  $\rho : U \rightarrow \mathfrak{h}_{\text{reg}}$  be the map defined on the coordinates  $\{x_B\}$  by

$$x_B = \prod_{B \subseteq C \in \mathcal{F}} u_C$$

Then  $\rho$  is birational with inverse

$$u_B = \begin{cases} x_B & \text{if } B \text{ is maximal in } \mathcal{F} \\ x_B/x_{c(B)} & \text{otherwise} \end{cases}$$

For any  $\alpha \in R_+$  set

$$P_{\alpha} = \frac{\alpha}{x_{p(\alpha)}}$$

**Lemma.**

(i) For any simple root  $\alpha$  such that  $B = p(\alpha)$ , we have

$$\alpha = x_B - \sum_{\substack{C \in \mathcal{F} \\ c(C)=B}} x_C \quad \text{and} \quad P_{\alpha} = 1 - \sum_{\substack{C \in \mathcal{F} \\ c(C)=B}} u_C \quad (3.2)$$

(ii) If  $\gamma \in R_+$  is the sum of two positive roots  $\alpha, \beta \in R_+$  with  $p(\alpha) = A$ ,  $p(\beta) = B$ , then for one of these, say  $\alpha$ , one has  $p(\gamma) = p(\alpha)$ ,  $p(\beta) = B \subsetneq p(\gamma) = A$ , and

$$P_{\gamma} = P_{\alpha} + P_{\beta} \prod_{B \subseteq C \subsetneq A} u_C \quad (3.3)$$

More precisely, if  $\gamma = \sum_{C \subseteq p(\gamma)} m_C \alpha_C$ , then

$$P_{\gamma} = \sum_{C \subseteq p(\gamma)} m_C \prod_{C \subseteq E \subsetneq p(\gamma)} u_E P_{\alpha_C} \quad (3.4)$$

*Proof.* (i) The formula (3.2) follows by direct computation. Namely, for any simple root  $\alpha$ ,

$$x_{p(\alpha)} P_{\alpha} = \alpha = x_{p(\alpha)} - \sum_{\substack{C \in \mathcal{F} \\ c(C)=B}} x_C$$

Since  $u_C = x_C/x_{c(C)}$ , one has

$$P_{\alpha} = 1 - \sum_{\substack{C \in \mathcal{F} \\ c(C)=B}} u_C$$



(ii) One has

$$P_\gamma = \frac{\gamma}{x_{p(\gamma)}} = \sum_{C \subseteq p(\gamma)} m_C \alpha_C \frac{1}{x_{p(\gamma)}} = \sum_{C \subseteq p(\gamma)} m_C \frac{\alpha_C}{x_C} \frac{x_C}{x_{p(\gamma)}}$$

Formula (3.4) follows by

$$P_{\alpha_C} = \frac{\alpha_C}{x_{p(\alpha_C)}} = \frac{\alpha_C}{x_C} \quad \frac{x_C}{x_{p(\gamma)}} = \prod_{C \subseteq E \subseteq p(\gamma)} u_E$$

The proof of (3.3) is similar.  $\square$

3.4.4. *De Concini–Procesi fundamental solutions.* For any  $\mathcal{F} \in \text{Mns}(D)$  and  $B \in \mathcal{F}$ , set

$$R_B = \sum_{\substack{\alpha \in \mathbb{R}_+ \\ p(\alpha) = B}} \mathbf{t}_\alpha \quad \text{and} \quad \mathbf{t}_B = \sum_{\substack{\alpha \in \mathbb{R}_+ \\ \text{supp}(\alpha) \subseteq B}} \mathbf{t}_\alpha = \sum_{\substack{C \in \mathcal{F} \\ C \subseteq B}} R_C$$

and hence

$$\sum_{B \in \mathcal{F}} R_B d \log(x_B) = \sum_{B \in \mathcal{F}} \mathbf{t}_B d \log(u_B) \quad (3.5)$$

$$\prod_{B \in \mathcal{F}} u_B^{\mathbf{t}_B} = \prod_{B \in \mathcal{F}} u_B^{\sum_{C \subseteq B} R_C} = \prod_{C \in \mathcal{F}} \prod_{B \supseteq C} u_B^{R_C} = \prod_{C \in \mathcal{F}} x_C^{R_C} \quad (3.6)$$

**Theorem.** For any  $\mathcal{F} \in \text{Mns}(D)$ , let  $\mathcal{U}_{\mathcal{F}} \subset U$  be the complement of the zeros of the polynomials  $P_\alpha$ ,  $\alpha \in \mathbb{R}$ , and  $\mathcal{B} \subset \mathcal{U}_{\mathcal{F}}$  a simply connected set, containing the point  $P_{\mathcal{F}} = \cap_{B \in \mathcal{F}} \{u_B = 0\}$ . There exists a unique holomorphic function  $H_{\mathcal{F}}$  on  $\mathcal{B}$ , such that  $H_{\mathcal{F}}(P_{\mathcal{F}}) = 1$  and, for every determination of  $\log(x_B)$ ,  $B \in \mathcal{F}$ , the multivalued function

$$G_{\mathcal{F}} = H_{\mathcal{F}} \prod_{B \in \mathcal{F}} x_B^{R_B} = H_{\mathcal{F}} \prod_{B \in \mathcal{F}} u_B^{\mathbf{t}_B}$$

is a solution of the holonomy equation  $dG = AG$ , where  $A = \sum_{\alpha} \mathbf{t}_\alpha d \log(\alpha)$ .  $H_{\mathcal{F}}$  is the unique solution of the differential equation

$$dH = [A_{\mathcal{F}}, H] + B_{\mathcal{F}} H \quad (3.7)$$

with the given initial condition, where

$$A_{\mathcal{F}} = \sum_{B \in \mathcal{F}} R_B d \log(x_B) = \sum_{B \in \mathcal{F}} \mathbf{t}_B d \log(u_B) \quad (3.8)$$

and

$$B_{\mathcal{F}} = A - A_{\mathcal{F}} = \sum_{\alpha \in \mathbb{R}_+} \mathbf{t}_\alpha d \log(P_\alpha) \quad (3.9)$$

which is holomorphic around in  $\mathcal{B}$ .

3.4.5. We start the proof of the theorem with the following

**Lemma.** For any  $B \in \mathcal{F}$ ,

$$[\mathbf{t}_B, B_{\mathcal{F}}]_{u_B=0} = 0$$

*Proof.* For any  $\alpha \in \mathbb{R}_+$ ,

$$\alpha = x_{p(\alpha)} P_\alpha = \prod_{B \supseteq p(\alpha)} u_B P_\alpha$$

It follows

$$d\log(\alpha) = \sum_{B \supseteq p(\alpha)} \frac{du_B}{u_B} + d\log(P_\alpha)$$

One has

$$\begin{aligned} A &= \sum_{\alpha \in \mathbb{R}_+} \mathbf{t}_\alpha d\log(\alpha) = \\ &= \sum_{B \in \mathcal{F}} \sum_{\substack{\alpha \in \mathbb{R}_+ \\ p(\alpha) \subseteq B}} \mathbf{t}_\alpha \frac{du_B}{u_B} + B_{\mathcal{F}} = \\ &= \sum_{B \in \mathcal{F}} \frac{\mathbf{t}_B}{u_B} du_B + B_{\mathcal{F}} \end{aligned}$$

Write  $B_{\mathcal{F}} = \sum_{B \in \mathcal{F}} P_B du_B$ . Then  $A \wedge A = 0$ , implies

$$\left[ \frac{\mathbf{t}_B}{u_B} + P_B, \frac{\mathbf{t}_C}{u_C} + P_C \right] = 0 \quad (3.10)$$

Multiplying by  $u_B$ , and then setting  $u_B = 0$ , one gets

$$\left[ \mathbf{t}_B, \frac{\mathbf{t}_C}{u_C} + P_C \right]_{u_B=0} = 0 \quad (3.11)$$

Since, for every  $B, C \in \mathcal{F}$ ,  $[\mathbf{t}_B, \mathbf{t}_C] = 0$ , (3.11) implies

$$[\mathbf{t}_B, P_C]_{u_B=0} = 0 \quad \text{and} \quad [\mathbf{t}_B, B_{\mathcal{F}}]_{u_B=0} = 0 \quad (3.12)$$

□

3.4.6. *Proof of Theorem 3.4.4.* It is enough to prove that there exists a unique solution  $H$  of the equation (3.7) with initial condition  $H(P_{\mathcal{F}}) = 1$ .

For  $m = 0$ ,  $\mathcal{U}_{\mathcal{F}}^{(0)} = U$ ,  $H^{(0)}$  is the constant function 1, and we can choose  $\mathcal{B}^{(0)} = U$ . Now let  $m > 0$  and let  $\mathcal{U}_{\mathcal{F}}^{(k)}$  be the complement of  $\bigcup_{\alpha \in B_m} \{P_\alpha = 0\}$  in  $U$ . Let  $\mathcal{B}^{(m)}$  any simply connected open set in  $\mathcal{B}^{(m-1)} \cap \mathcal{U}_{\mathcal{F}}^{(m)}$  containing  $P_{\mathcal{F}}$ .

We assume that  $H^{(m)} = \sum_{k \geq 0} H_k$ ,  $H_k$  being of degree  $k$  in  $\mathbf{t}_{\mathbb{R}}$ , with  $H_0(P_{\mathcal{F}}) = 1$  and  $H_k(P_{\mathcal{F}}) = 0$ ,  $k > 0$ . Equation (3.7) is then equivalent to the recursive system

$$dH_{k+1} = [A_{\mathcal{F}}, H_k] + B_{\mathcal{F}} H_k \quad (3.13)$$

and  $dH_0 = 0$ , with the given initial conditions.  $dH_0 = 0$  implies  $H_0 = 1$  and  $dH_1 = B_{\mathcal{F}} H_0$ . The form  $B_{\mathcal{F}}$  is holomorphic around  $P_{\mathcal{F}}$ .

Assume by induction that  $[A_{\mathcal{F}}, H_k] + B_{\mathcal{F}} H_k$  is holomorphic. We claim that, choosing the solution of (3.13)  $H^{(k+1)}$  with  $H_{k+1}(P_{\mathcal{F}}) = 0$ ,  $[A_{\mathcal{F}}, H^{k+1}]$  is again holomorphic. To see this it is enough to show that  $[\mathbf{t}_B, H^{k+1}]$  vanishes on  $u_B = 0$ . Assume by induction that  $[\mathbf{t}_B, H^k]_{u_B=0} = 0$  for all  $B \in \mathcal{F}$ . Then we have

$$[\mathbf{t}_B, dH^{k+1}] = [\mathbf{t}_B, [A_{\mathcal{F}}, H_k]] + [\mathbf{t}_B, B_{\mathcal{F}}] H_k + B_{\mathcal{F}} [\mathbf{t}_B, H_k]$$

By induction and Lemma 3.4.5, all terms vanish on  $u_B = 0$ .  $[A_{\mathcal{F}}, H_{k+1}]$  is again holomorphic, and the result follows.

3.4.7. *Properties of the De Concini–Procesi associators.* Let  $\mathcal{F}, \mathcal{G} \in \text{Mns}(D)$  be two fundamental maximal nested sets and let  $G_{\mathcal{F}}, G_{\mathcal{G}}$  be the two associated solutions. By the general theory, comparing  $G_{\mathcal{F}}, G_{\mathcal{G}}$  on the fundamental Weyl chamber, there is an invertible multiplicative constant, which we denote by  $\Upsilon_{\mathcal{F}\mathcal{G}}$ , such that

$$G_{\mathcal{G}} = G_{\mathcal{F}} \Upsilon_{\mathcal{F}\mathcal{G}}$$

We refer to the constants  $\Upsilon_{\mathcal{F}\mathcal{G}}$  as the De Concini–Procesi associators.

**Theorem.** *The De Concini–Procesi associators satisfy the following properties:*

- (i) **Orientation:**  $\Upsilon_{\mathcal{F}\mathcal{G}} = \Upsilon_{\mathcal{G}\mathcal{F}}^{-1}$ .
- (ii) **Transitivity:**  $\Upsilon_{\mathcal{F}\mathcal{G}} = \Upsilon_{\mathcal{F}\mathcal{H}} \Upsilon_{\mathcal{H}\mathcal{G}}$ .
- (iii) **Support:**  $\Upsilon_{\mathcal{F}\mathcal{G}} \in \mathfrak{t}_{\text{supp}(\mathcal{F}, \mathcal{G})}$ .
- (iv) **Central Support:**  $\Upsilon_{\mathcal{F}\mathcal{G}}$  commutes with  $\{\mathfrak{t}_{\alpha} \mid \alpha \in \mathbf{R}_{\mathfrak{z}\text{supp}(\mathcal{F}, \mathcal{G})}\}$ .
- (v) **Forgetfulness:** For any equivalent elementary pairs  $(\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}')$ ,  $\Upsilon_{\mathcal{F}\mathcal{G}} = \Upsilon_{\mathcal{F}'\mathcal{G}'}$ .

*Proof.* (i) – (ii) The properties of orientation and transitivity follow directly from the definition of  $\Upsilon_{\mathcal{F}\mathcal{G}}$ .

(iii) – (iv) It is enough to prove the properties of support and central support for all  $\mathcal{F}, \mathcal{G}$  which differ by only one element. We can assume there is a subdiagram  $B = \text{supp}(\mathcal{F}, \mathcal{G})$  and two vertices  $i, j \in B$  such that  $\mathfrak{z}\text{supp}(\mathcal{F}, \mathcal{G}) = B \setminus \{i, j\}$ , i.e.,  $\mathcal{F} = \mathcal{U} \cup \{B \setminus \{i\}\}$  and  $\mathcal{G} = \mathcal{U} \cup \{B \setminus \{j\}\}$ , where  $\mathcal{U} = \mathcal{F} \cap \mathcal{G}$ .

Let  $\{u_C^{\mathcal{F}}\}_{C \in \mathcal{F}}, \{u_C^{\mathcal{G}}\}_{C \in \mathcal{G}}$  be two sets of coordinates and  $R_C^{\mathcal{F}}, R_C^{\mathcal{G}}$  two residues. Then  $u_C^{\mathcal{F}} = u_C^{\mathcal{G}}$  and  $R_C^{\mathcal{F}} = R_C^{\mathcal{G}}$  for every  $C \in \mathcal{U} \setminus \{B\}$ , and

$$R_{B \setminus \{i\}}^{\mathcal{F}} + R_B^{\mathcal{F}} = R_{B \setminus \{j\}}^{\mathcal{G}} + R_B^{\mathcal{G}} =: K$$

as it corresponds with the sum of all  $\mathfrak{t}_{\alpha}$  such that  $\text{supp}(\alpha)$  is contained in  $B$  but not in  $B \setminus \{i, j\}$ . Set  $R_i = R_{B \setminus \{i\}}^{\mathcal{F}}, R_j = R_{B \setminus \{j\}}^{\mathcal{G}}, x_i = x_{B \setminus \{i\}}, x_j = x_{B \setminus \{j\}}, u_i = u_{B \setminus \{i\}}^{\mathcal{F}}$ , and  $u_j = u_{B \setminus \{j\}}^{\mathcal{G}}$ . Then we have

$$\begin{aligned} G_{\mathcal{F}} &= H_{\mathcal{F}} \prod_{C \in \mathcal{F}} x_C^{R_C^{\mathcal{F}}} = H_{\mathcal{F}} \prod_{C \in \mathcal{U} \setminus B} x_C^{R_C} x_B^{R_B^{\mathcal{F}}} x_i^{R_i} = H_{\mathcal{F}} \prod_{C \in \mathcal{U} \setminus B} x_C^{R_C} x_B^K u_i^{R_i} \\ G_{\mathcal{G}} &= H_{\mathcal{G}} \prod_{C \in \mathcal{G}} x_C^{R_C^{\mathcal{G}}} = H_{\mathcal{G}} \prod_{C \in \mathcal{U} \setminus B} x_C^{R_C} x_B^{R_B^{\mathcal{G}}} x_j^{R_j} = H_{\mathcal{G}} \prod_{C \in \mathcal{U} \setminus B} x_C^{R_C} x_B^K u_j^{R_j} \end{aligned}$$

The functions  $H_{\mathcal{F}} u_i^{R_i}, H_{\mathcal{G}} u_j^{R_j}$  satisfy the same equation

$$dF = AF - FA_{\mathfrak{z}} \quad \text{where} \quad A_{\mathfrak{z}} = K d \log(x_B) + \sum_{C \in \mathcal{U} \setminus \{B\}} R_C d \log(x_C)$$

Restricted to  $\{u_C = 0\}_{C \in \mathcal{U}}$ , the two functions satisfy the same holonomy equation, and therefore they differ by a constant  $\Upsilon$  in the algebra generated by the residues of this equation, which commute with the elements  $K$  and  $\{R_C\}_{C \in \mathcal{U} \setminus \{B\}}$ . It follows that the same is true for  $G_{\mathcal{F}}$  and  $G_{\mathcal{G}}$ , and  $\Upsilon_{\mathcal{F}\mathcal{G}} = \Upsilon$ .  $\square$

**Corollary.** *The collection of De Concini–Procesi associators defines a weak quasi-Coxeter structure on the holonomy algebra  $\mathfrak{t}_{\mathbf{R}}$ .*

**3.5. The differential quasi-Coxeter structure.** The notion of weak quasi-Coxeter structure is similarly defined for the algebra  $\mathcal{U}$  and the category  $\mathcal{O}$ . Moreover, any weak quasi-Coxeter structure on the holonomy algebra  $\mathfrak{t}_{\mathbb{R}}$  induces a weak quasi-Coxeter structure on  $\mathcal{U}$  and  $\mathcal{O}$  by the mean of the morphism

$$\xi_{\mathbb{R}} : \mathfrak{t}_{\mathbb{R}} \rightarrow \mathcal{U}$$

The correction of the monodromy performed in 2.7 allows to extend the induced structure to a quasi-Coxeter structure.

**Theorem.** *Set  $\hbar = \pi \iota \hbar$ . Then,*

- (i) *The associators  $\Upsilon_{\mathcal{FG}}$  and local monodromies*

$$S_{i,C}^{\nabla} = \tilde{s}_i \cdot \exp(\hbar/2 \cdot C_{\alpha_i}) \quad (3.14)$$

*endow the category  $\mathcal{O}$  with a quasi-Coxeter structure  $\mathcal{Q}_{\kappa}^{\nabla}$  of type  $D$ .*

- (ii) *For any  $V \in \mathcal{O}$  and maximal nested set  $\mathcal{F}$ , the representation*

$$\pi_{\mathcal{F}} : B_W \rightarrow GL(V[[\hbar]])$$

*obtained from the quasi-Coxeter structure  $\mathcal{Q}_{\kappa}^{\nabla}$  coincides with the monodromy of  $(\tilde{\nabla}, \tilde{\nabla}_{\kappa})$  expressed in the fundamental solution  $\Psi_{\mathcal{F}}$ .*

#### 4. A DIFFERENTIAL BRAIDED QUASI-COXETER STRUCTURE ON CATEGORY $\mathcal{O}$

In this section, we review the notion of *braided* quasi-Coxeter category [1]. We will prove in Section 5 that the quasi-Coxeter structure on  $\mathcal{O}^{\text{int}}$  which arises from the monodromy of the Casimir connection is part of a braided quasi-Coxeter one which also gives rises to the monodromy of the KZ equations for  $\mathfrak{g}$  and all its Levi subalgebras. Similarly to its non-braided counterpart, the latter structure is universal in that it can be defined on a double holonomy algebra  $\mathfrak{t}_{\mathbb{R},n}$  generated by the algebra  $\mathfrak{t}_{\mathbb{R}}$  introduced in 3.3 and the holonomy algebra  $\mathfrak{t}_n$  of the pure braid group on  $n$  strands. The algebra  $\mathfrak{t}_{\mathbb{R},n}$  is introduced and studied in this section.

##### 4.1. Braided quasi-Coxeter categories.

###### 4.1.1. Strict $D$ -monoidal categories.

**Definition.** A *strict  $D$ -monoidal category*  $\mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\}, \{J_{BB'}\})$  is a  $D$ -category  $\mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\})$  where

- for any  $B \subseteq D$ ,  $(\mathcal{C}_B, \otimes_B)$  is a strict monoidal category
- for any  $B \subseteq B'$ , the functor  $F_{BB'}$  is endowed with a tensor structure  $J_{BB'}$  with the additional condition that, for every  $B \subseteq B' \subseteq B''$ ,  $J_{BB'} \circ J_{B'B''} = J_{BB''}$ .

###### 4.1.2. $D$ -monoidal categories.

**Definition.** A  *$D$ -monoidal category*

$$\mathcal{C} = (\{(\mathcal{C}_B, \otimes_B, \Phi_B)\}, \{F_{BB'}\}, \{J_{BB'}^{\mathcal{F}}\})$$

is the datum of

- A  $D$ -category  $(\{\mathcal{C}_B\}, \{F_{BB'}\})$  such that each  $(\mathcal{C}_B, \otimes_B, \Phi_B)$  is a tensor category, with  $\mathcal{C}_{\emptyset}$  a strict tensor category, *i.e.*,  $\Phi_{\emptyset} = \text{id}$ .
- for any pair  $B \subseteq B'$  and  $\mathcal{F} \in \text{Mns}(B, B')$ , a tensor structure  $J_{\mathcal{F}}^{BB'}$  on the functor  $F_{BB'} : \mathcal{C}_{B'} \rightarrow \mathcal{C}_B$

with the additional condition that, for any  $B \subseteq B' \subseteq B''$ ,  $\mathcal{F} \in \text{Mns}(B'', B')$ ,  $\mathcal{G} \in \text{Mns}(B', B)$ ,

$$J_{BB'}^{\mathcal{G}} \circ J_{B'B''}^{\mathcal{F}} = J_{BB''}^{\mathcal{F} \cup \mathcal{G}}$$

4.1.3. *Braided  $D$ -monoidal categories.*

**Definition.** A *braided  $D$ -monoidal category*

$$\mathcal{C} = (\{(\mathcal{C}_B, \otimes_B, \Phi_B, \beta_B)\}, \{(F_{BB'}, J_{BB'}^{\mathcal{F}})\})$$

is the datum of

- a  $D$ -monoidal category  $(\{(\mathcal{C}_B, \otimes_B, \Phi_B)\}, \{(F_{BB'}, J_{BB'}^{\mathcal{F}})\})$
- for every  $B \subseteq D$ , a commutativity constraint  $\beta_B$  in  $\mathcal{C}_B$ , defining a braiding in  $(\mathcal{C}_B, \otimes_B, \Phi_B)$ .

**Remark.** Note that the tensor functors  $(F_{BB'}, J_{BB'}^{\mathcal{F}}) : \mathcal{C}_{B'} \rightarrow \mathcal{C}_B$  are *not* assumed to map the commutativity constraint  $\beta_{B'}$  to  $\beta_B$ .

4.1.4. *Braided quasi-Coxeter categories.*

**Definition.** A *quasi-Coxeter braided monoidal category of type  $D$*

$$\mathcal{C} = (\{(\mathcal{C}_B, \otimes_B, \Phi_B, \beta_B)\}, \{(F_{BB'}, J_{BB'}^{\mathcal{F}})\}, \{\Upsilon_{\mathcal{F}\mathcal{G}}\}, \{S_i\})$$

is the datum of

- a quasi-Coxeter category of type  $D$ ,

$$\mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\}, \{\Upsilon_{\mathcal{F}\mathcal{G}}\}, \{S_i\})$$

- a braided  $D$ -monoidal category

$$\mathcal{C} = (\{(\mathcal{C}_B, \otimes_B, \Phi_B, \beta_B)\}, \{(F_{BB'}, J_{\mathcal{F}}^{BB'})\})$$

satisfying the following conditions

- for any  $B \subseteq B'$ , and  $\mathcal{G}, \mathcal{F} \in \text{Mns}(B, B')$ , the natural transformation  $\Upsilon_{\mathcal{F}\mathcal{G}} \in \text{Aut}(F_{BB'})$  determines an isomorphism of tensor functors  $(F_{BB'}, J_{BB'}^{\mathcal{G}}) \rightarrow (F_{BB'}, J_{BB'}^{\mathcal{F}})$ , that is for any  $V, W \in \mathcal{C}_{B'}$ ,

$$(\Upsilon_{\mathcal{G}\mathcal{F}})_{V \otimes W} \circ (J_{BB'}^{\mathcal{F}})_{V, W} = (J_{BB'}^{\mathcal{G}})_{V, W} \circ ((\Upsilon_{\mathcal{G}\mathcal{F}})_V \otimes (\Upsilon_{\mathcal{G}\mathcal{F}})_W)$$

- for any  $i \in D$ , the following holds:

$$\Delta_{J_i}(S_i) = (R_i)_{J_i}^{21} \cdot (S_i \otimes S_i)$$

## 4.2. The KZ-holonomy algebra.

4.2.1. *Cosimplicial structure on  $\mathcal{U}$ .* As before, let  $\mathcal{O}$  be the category of deformation highest weight integrable  $\mathfrak{g}$ -modules,  $\mathcal{A}$  the category of topologically free  $\mathbb{C}[[\hbar]]$ -modules, and  $\mathbf{f} : \mathcal{O} \rightarrow \mathcal{A}$  the forgetful functor. Set  $\mathcal{U}^n = \text{End}(\mathbf{f}^{\boxtimes n})$ , where  $\mathbf{f}^{\boxtimes 1} = \mathbf{f}$ , and  $\mathbf{f}^{\boxtimes n} : \mathcal{O}^n \rightarrow \mathcal{A}$ ,  $\mathbf{f}^{\boxtimes n}(V_1, \dots, V_n) = V_1 \otimes \dots \otimes V_n$ .

The tower of algebras  $\{\mathcal{U}^n\}_{n \geq 0}$  is a cosimplicial complex of algebras

$$\mathbf{k} \rightrightarrows \text{End}(\mathbf{f}) \rightrightarrows \text{End}(\mathbf{f}^{\boxtimes 2}) \rightrightarrows \text{End}(\mathbf{f}^{\boxtimes 3}) \quad \dots$$

with face morphisms  $d_n^i : \text{End}(\mathbf{f}^{\boxtimes n}) \rightarrow \text{End}(\mathbf{f}^{\boxtimes n+1})$ ,  $i = 0, \dots, n+1$ , given by

$$(d_0^0 \varphi)_X : \mathbf{f}(X) \longrightarrow \mathbf{f}(X) \otimes \mathbf{1} \xrightarrow{1 \otimes \varphi} \mathbf{f}(X) \otimes \mathbf{1} \longrightarrow \mathbf{f}(X)$$

$$(d_0^1 \varphi)_X : \mathbf{f}(X) \longrightarrow \mathbf{1} \otimes \mathbf{f}(X) \xrightarrow{\varphi \otimes 1} \mathbf{1} \otimes \mathbf{f}(X) \longrightarrow \mathbf{f}(X)$$

where  $\mathbf{1}$  is the trivial module,  $X \in \mathcal{O}$ ,  $\varphi \in \mathfrak{k}$ , and

$$(d_n^i \varphi)_{X_1, \dots, X_{n+1}} = \begin{cases} \text{id} \otimes \varphi_{X_2, \dots, X_{n+1}} & i = 0 \\ \varphi_{X_1, \dots, X_i} \otimes \varphi_{X_{i+1}, \dots, X_{n+1}} & 1 \leq i \leq n \\ \varphi_{X_1, \dots, X_n} \otimes \text{id} & i = n+1 \end{cases}$$

for  $\varphi \in \text{End}(\mathfrak{f}^{\boxtimes n})$ ,  $X_i \in \mathcal{O}$ ,  $i = 1, \dots, n+1$ . The degeneration homomorphisms  $s_n^i : \text{End}(\mathfrak{f}^{\boxtimes n}) \rightarrow \text{End}(\mathfrak{f}^{\boxtimes n-1})$ , for  $i = 1, \dots, n$ , are

$$(s_n^i \varphi)_{X_1, \dots, X_{n-1}} = \varphi_{X_1, \dots, X_{i-1}, \mathbf{1}, X_i, \dots, X_{n-1}}$$

**4.2.2. The holonomy algebra of the KZ-connection.** Let  $\mathfrak{t}_n$  be the algebra generated over  $\mathbb{C}$  by the elements  $\{\mathfrak{t}_{ij}\}$ ,  $1 \leq i < j \leq n$  with relations

$$[\mathfrak{t}_{ij}, \mathfrak{t}_{ik} + \mathfrak{t}_{jk}] = 0 \quad [\mathfrak{t}_{ij}, \mathfrak{t}_{kl}] = 0$$

for any  $i, j, k, l$  such that  $\{i, j\} \cap \{k, l\} = \emptyset$ .

**4.2.3. The cosimplicial structure on  $\{\mathfrak{t}_n\}_{n \geq 0}$ .** The algebras  $\mathfrak{t}_n$  are naturally endowed with a cosimplicial structure. The insertion coproduct maps

$$d_n^k : \mathfrak{t}_n \rightarrow \mathfrak{t}_{n+1} \quad k = 0, 1, \dots, n+1$$

are defined by

$$d_n^k(\mathfrak{t}_{ij}) = \delta_{ki}(\mathfrak{t}_{ij} + \mathfrak{t}_{i+1,j}) + \delta_{kj}(\mathfrak{t}_{ij} + \mathfrak{t}_{i,j+1}) \quad k = 1, \dots, n$$

and

$$d_n^0(\mathfrak{t}_{ij}) = \mathfrak{t}_{i+1,j+1} \quad d_n^{n+1}(\mathfrak{t}_{ij}) = \mathfrak{t}_{ij}$$

The degeneration homomorphisms  $s_n^k : \mathfrak{t}_n \rightarrow \mathfrak{t}_{n-1}$ ,  $k = 1, \dots, n$  are

$$s_n^k(\mathfrak{t}_{ij}) = (1 - \delta_{ki} - \delta_{kj})\mathfrak{t}_{ij}$$

**4.2.4. The holonomy algebra and category  $\mathcal{O}$ .** Let  $r \in \mathcal{U}^2$  be the classical  $r$ -matrix. The following proposition is well-known [6].

**Proposition.** *The linear map  $\xi_n : \mathfrak{t}_n \rightarrow \mathcal{U}^n$  defined by*

$$\xi_n(\mathfrak{t}_{ij}) = \mathfrak{h} \cdot (r^{ij} + r^{ji})$$

*is a morphism of algebras, compatible with cosimplicial structure and the natural gradation on  $\mathfrak{t}_n$ <sup>1</sup>.*

### 4.3. The double holonomy algebra.

**4.3.1. A  $R$ -refinement of  $\mathfrak{t}_n$ .** We now defined an extended version of the holonomy algebra, which naturally embed into  $\mathcal{U}^n$ .

**Definition.** Let  $R$  be a fixed root system. The  $R$ -holonomy algebra  $\mathfrak{t}'_{R,n}$  is the algebra over  $\mathbb{C}$  generated by the symbols  $\{\Omega_{ij}, \Omega_{ij}^\alpha\}$ ,  $1 \leq i < j \leq n$ ,  $\alpha \in R \cup \{0\}$  with relations

$$[\Omega_{ij}, \Omega_{ik} + \Omega_{jk}] = 0 \quad [\Omega_{ij}, \Omega_{kl}] = 0 \quad (4.1)$$

and

$$[\Omega_{ij}^\alpha, \Omega_{kl}^\beta] = 0 \quad \Omega_{ij} = \sum_{\alpha \in R \cup 0} \Omega_{ij}^\alpha \quad (4.2)$$

for any  $i, j, k, l$  such that  $\{i, j\} \cap \{k, l\} = \emptyset$  and  $\alpha, \beta \in R \cup \{0\}$ .<sup>2</sup> Finally, we assume  $[\Omega_{ij}^0, \Omega_{kl}^0] = 0$  for every  $i, j, k, l$ .

<sup>1</sup>As in  $\mathfrak{t}_R$ ,  $\deg \mathfrak{t}_{ij} = 1$ .

<sup>2</sup>If  $|R| = +\infty$ , then the relation (4.2) is to be intended as in 3.3.

The tower of algebras  $\{\mathfrak{t}'_{R,n}\}_{n \geq 0}$  is naturally endowed with a cosimplicial structure, which extends the cosimplicial structure on  $\mathfrak{t}_n$ . in particular,

$$d_n^k(\Omega_{ij}^\alpha) = \delta_{ki}(\Omega_{ij}^\alpha + \Omega_{i+1,j}^\alpha) + \delta_{kj}(\Omega_{ij}^\alpha + \Omega_{i,j+1}^\alpha) \quad k = 1, \dots, n$$

Similarly for the degeneration maps.

**Proposition.** *The linear map  $\xi : \mathfrak{t}'_{R,n} \rightarrow \mathcal{U}^n$  defined by*

$$\xi'_{R,n}(\Omega_{ij}) = \mathfrak{h} \cdot r^{ij} \quad \xi'_{R,n}(\Omega_{ij}^\alpha) = \mathfrak{h} \cdot r_\alpha^{ij}$$

*is a morphism of algebras, compatible with the cosimplicial structure and the natural grading on  $\mathfrak{t}'_{R,n}$ .*

Concretely, let  $\mathfrak{g}$  be the Kac–Moody algebra associated with  $R$ . Then

$$r_\alpha^{ij} = (e_\alpha)_a^{(i)} \otimes (e_{-\alpha})^{a,(j)}$$

where  $\{(e_\alpha)_a\}, \{(e_{-\alpha})^a\}$  are dual basis of  $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$  and

$$\Omega_{ij}^0 = x_a^{(i)} \otimes x^{a,(j)}$$

where  $\{x_a\}, \{x^a\}$  are dual basis of  $\mathfrak{h}$ .

There is a natural action of  $\mathfrak{h}^{\otimes n}$  on  $\mathfrak{t}_n$ , i.e., for any  $h \in \mathfrak{h}$

$$h^{(k)} \cdot \Omega_{ij}^\alpha = (\delta_{ki} - \delta_{kj})\alpha(h)\Omega_{ij}^\alpha$$

**4.3.2. Double holonomy algebra.** We now extend the set of generators in order to describe simultaneously the  $n$ th–Casimir connection.

**Definition.** Let  $\mathfrak{t}_{R,n}$  be the algebra generated over  $\mathbb{C}$  by the elements  $\{\Omega_{ij}, \Omega_{ij}^\alpha\}$  with the relations (4.1), (4.2), and by the elements  $K_\alpha^{(n)}$  and  $K_{\alpha,k}$ ,  $\alpha > 0$ ,  $k = 1, \dots, n$ , with the relations

$$[K_\alpha^{(n)}, \sum_{\beta \in \Psi} K_\beta^{(n)}] = 0$$

for any subsystem  $\Psi \subset R$ ,  $\text{rk}(\Psi) = 2$ , and  $\alpha \in \Psi$ . Finally, we add the relations

$$[\Omega_{ij}, K_\alpha^{(n)}] = 0$$

for all  $1 \leq i, j \leq n$ ,  $\alpha \in R_+$ ,

$$[\Omega_{ij}^\alpha, K_{\beta,k}] = 0$$

if  $k \notin \{i, j\}$ , and

$$K_\alpha^{(n)} = \sum_{i < j} \Omega_{ij}^\alpha + \Omega_{ij}^{-\alpha} + \sum_{k=1}^n K_{\alpha,k} \quad (4.3)$$

**4.3.3. Weight decomposition.** The elements  $K_\alpha^{(n)}$  should be thought of concretely in  $\mathcal{U}^n$  as  $\frac{\mathfrak{h}}{2}\Delta^{(n)}(\mathcal{K}_\alpha^+)$  where  $\mathcal{K}_\alpha^+$  is the Wick–ordered (truncated) Casimir element.

The action of  $\mathfrak{h}^{\otimes n}$  on  $\mathfrak{t}_{R,n}$  extends to the elements  $K_\alpha^{(n)}$ . Namely, in  $\mathcal{U}^n$ , we have

$$\begin{aligned} \Delta^{(n)}(\mathcal{K}_\alpha^+) &= 2\Delta^{(n)}(e_{-\alpha}e_\alpha) = \\ &= 2 \left( \sum_{i < j} \Omega_{ij}^\alpha + \Omega_{ij}^{-\alpha} + \mathcal{K}_\alpha^0 \right) \end{aligned}$$

where  $\mathcal{K}_\alpha^0 = \sum_{i=1}^n (1^{\otimes k-1} \otimes \mathcal{K}_\alpha^+ \otimes 1^{\otimes n-k})$  is a weight zero element. Therefore we define the action of  $\mathfrak{h}$  by

$$h^{(k)} \cdot K_{\alpha,l} = 0$$

and

$$h^{(k)} \cdot K_\alpha^{(n)} = h^{(k)} \cdot \left( \sum_{i < j} \Omega_{ij}^\alpha + \Omega_{ij}^{-\alpha} \right)$$

consistently with the relation (4.3).

4.3.4. *The cosimplicial structure on  $\{\mathfrak{t}_{R,n}\}_{n \geq 0}$ .* The algebras  $\mathfrak{t}_{R,n}$  are naturally endowed with a cosimplicial structure. Set

$$K_{\alpha,k}^{(m)} = \sum_{k \leq i < j \leq m+k-1} \Omega_{ij}^\alpha + \Omega_{ij}^{-\alpha} + \sum_{l=k}^{m+k-1} K_{\alpha,l}$$

so that  $K_\alpha^{(n)} = K_{\alpha,1}^{(n)}$  and  $K_{\alpha,i} = K_{\alpha,i}^{(1)}$ . The insertion coproduct maps

$$d_n^k : \mathfrak{t}_{R,n} \rightarrow \mathfrak{t}_{R,n+1} \quad k = 0, 1, \dots, n+1$$

are defined on  $\Omega_{ij}, \Omega_{ij}^\alpha$  as in the case of  $\mathfrak{t}'_{R,n}$ , and on  $K_{\alpha,i}^{(m)}$  by

$$d_n^k(K_{\alpha,i}^{(m)}) = \begin{cases} K_{\alpha,i+1}^{(m)} & \text{if } k < i \\ K_{\alpha,i}^{(m+1)} & \text{if } k = i, \dots, m+i \\ K_{\alpha,i}^{(m)} & \text{if } k = m+i+1, \dots, n \end{cases}$$

for  $k = 1, \dots, n$ , and

$$d_n^0(K_{\alpha,i}^{(m)}) = K_{\alpha,i+1}^{(m)} \quad d_n^{n+1}(K_{\alpha,i}^{(m)}) = K_{\alpha,i}^{(m)}$$

The degeneration homomorphisms  $s_n^k : \mathfrak{t}_{R,n} \rightarrow \mathfrak{t}_{R,n-1}$ ,  $k = 1, \dots, n$  are similarly defined. In particular,

$$s_n^k(K_{\alpha,i}^{(m)}) = \begin{cases} K_{\alpha,i-1}^{(m)} & \text{if } k < i \\ K_{\alpha,i}^{(m-1)} & \text{if } k = i, \dots, m+i \\ K_{\alpha,i}^{(m)} & \text{if } k = m+i+1, \dots, n \end{cases}$$

4.3.5. *Double holonomy algebra and category  $\mathcal{O}$ .*

**Proposition.** *The linear map  $\xi_{R,n} : \mathfrak{t}_{R,n} \rightarrow \mathcal{U}^n$  defined by*

$$\xi_{R,n}(\Omega_{ij}^\alpha) = \mathfrak{h} \cdot r_\alpha^{ij} \quad \xi_{R,n}(\Omega_{ij}^{-\alpha}) = \mathfrak{h} \cdot r_\alpha^{ji}$$

and

$$\xi_{R,n}(K_\alpha^{(n)}) = \frac{\mathfrak{h}}{2} \cdot \Delta^{(n)}(\mathcal{K}_\alpha^+) \quad \xi_{R,n}(K_{\alpha,k}) = \frac{\mathfrak{h}}{2} \cdot (1^{\otimes k-1} \otimes \mathcal{K}_\alpha^+ \otimes 1^{\otimes n-k})$$

is a morphism of cosimplicial algebras, compatible with the action of  $\mathfrak{h}$  and the natural gradation on  $\mathfrak{t}_{R,n}$ <sup>1</sup>. In particular,

$$\xi_{R,n}(K_{\alpha,k}^{(m)}) = \frac{\mathfrak{h}}{2} (1^{\otimes k-1} \otimes \Delta^{(m)}(\mathcal{K}_\alpha^+) \otimes 1^{\otimes n-m-k+1})$$

---

<sup>1</sup>All generators have  $\deg = 1$ .



*Proof.* The relations satisfied by  $\Omega_{ij}$ ,  $1 \leq i < j \leq n$ , follow from the commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{t}_{R,n} & \xrightarrow{\xi} & \mathcal{U}^n \\ \uparrow & \nearrow \xi & \\ \mathfrak{t}_n & & \end{array}$$

The  $tt$ -relations

$$[\mathcal{K}_\alpha^{(n)}, \sum_{\beta \in \Psi} \mathcal{K}_\beta^{(n)}] = 0$$

are satisfied by the elements  $\mathcal{K}_\alpha^+ \in \mathcal{U}^n$  as in Prop. 7.1.1. The commutativity relations

$$[\Omega_{ij}, \mathcal{K}_\alpha^{(n)}] = 0$$

follow from the  $\mathfrak{g}$ -invariance of  $r^{ij} + r^{ji}$  in  $\mathcal{U}^n$ . The last statement is clear.  $\square$

#### 4.4. The differential braided quasi-Coxeter structure.

**Definition.** A *weak braided quasi-Coxeter structure* on  $\mathfrak{t}_R$  is the datum of

- (i) for each connected subdiagram  $B \subseteq D$ , an  $R$ -matrices  $R_B \in \mathfrak{t}_{B,2}$  and an associator  $\Phi_B \in \mathfrak{t}_{B,3}$ , which are of the following form

$$R_B = \exp\left(\frac{\Omega_B}{2}\right) \quad \text{and} \quad \Phi_B = \Phi'_B(\Omega_{B,12}, \Omega_{B,23})$$

where  $\Phi'_B$  is a Lie associator.

- (ii) for each pair of subdiagrams  $B' \subseteq B \subseteq D$  and maximal nested set  $\mathcal{F} \in \text{Mns}(B', B)$ , a relative twist  $J_{\mathcal{F}}^{BB'} \in \mathfrak{t}_{B',2}^{B'}$ , satisfying

$$J_{\mathcal{F}}^{BB'} = 1 \mod (\mathfrak{t}_{B,2})_{\geq 1} \tag{4.4}$$

$$(\Phi_B)_{J_{\mathcal{F}}^{BB'}} = \Phi_{B'} \tag{4.5}$$

<sup>1</sup> and satisfying the *factorization property*

$$J_{\mathcal{F}_1 \cup \mathcal{F}_2}^{BB''} = J_{\mathcal{F}_1}^{BB'} \cdot J_{\mathcal{F}_2}^{B'B''}$$

where  $B'' \subseteq B' \subseteq B$ ,  $\mathcal{F}_1 \in \text{Mns}(B, B')$  and  $\mathcal{F}_2 \in \text{Mns}(B', B'')$ .

- (iii) for any  $B' \subseteq B$  and pair of maximal nested sets  $\mathcal{F}, \mathcal{G} \in \text{Mns}(B, B')$ , an element  $\Upsilon_{\mathcal{F}\mathcal{G}} \in \mathfrak{t}_B$  such that

$$\Upsilon_{\mathcal{F}\mathcal{G}} = 1 \mod (\mathfrak{t}_B)_{\geq 1}$$

and satisfying the properties 3.3.2.

In the next section, we will prove the following

**Theorem.** *The weak quasi-Coxeter structure on  $\mathfrak{t}_R$  extends to a weak braided quasi-Coxeter structure.*

This induces a weak braided quasi-Coxeter structure on  $\mathcal{U}$ , and therefore on  $\mathcal{O}^{\text{int}}$ , which can be completed into a braided quasi-Coxeter one. In particular, we get

<sup>1</sup>For any element  $\Phi \in \mathfrak{t}_{R,3}$  and  $J \in \mathfrak{t}_{R,2}$ , we set

$$\Phi_J = d_2^3(J) \cdot d_2^1(J) \cdot \Phi \cdot d_2^2(J)^{-1} d_2^0(J)^{-1}$$

**Theorem.** *There exists a structure of braided quasi-Coxeter category on  $\mathcal{O}^{\text{int}}$ , which interpolates the monodromy of the KZ and the Casimir connections.*

## 5. DIFFERENTIAL TWISTS AND THE FUSION OPERATOR

This section follows [20, Sec. 3–7] closely, and adapts the construction of the differential twist and the fusion operator to the double holonomy algebra  $\mathfrak{t}_{\mathbf{R},2}$  introduced in 4.3. In particular, we introduce the notion of differential twist with values in  $\mathfrak{t}_{\mathbf{R},2}$ , and show that it induces a weak quasi-Coxeter structure on  $\mathfrak{t}_{\mathbf{R}}$ . We then show that a differential twist can be obtained by constructing an appropriate fusion operator with values in  $\mathfrak{t}_{\mathbf{R},2}$ .

**5.1. Differential twist.** Let  $\mathcal{C}_{\mathbf{R}} = \{t \in \mathfrak{h} \mid \alpha_i(t) > 0, \forall i \in \mathbf{I}\} \subset \mathfrak{h}_{\mathbf{R}}^{\text{e}}$  be the fundamental chamber of  $\mathfrak{h}$ , and set  $\mathcal{C} = \mathcal{C}_{\mathbf{R}} + i\mathfrak{h}_{\mathbf{R}}^{\text{e}}$ . Let  $\mathfrak{t}_{\mathbf{R},2}$  be the double holonomy algebra, and define  $\tilde{r} \in \mathfrak{t}_{\mathbf{R},2}$  by

$$\tilde{r} = \frac{1}{2} \sum_{\alpha \in \mathbf{R}_+} \Omega^{\alpha} - \Omega^{-\alpha}$$

**Definition.** A *differential twist* of type  $\mathbf{R}$  is a holomorphic map  $F = F_{\mathbf{R}} : \mathcal{C} \rightarrow \mathfrak{t}_{\mathbf{R},2}$  such that

- (i)  $s_2^1(F) = 1 = s_2^2(F)$ .
- (ii)  $F = 1 + f \pmod{(\mathfrak{t}_{\mathbf{R},2})_{\geq 2}}$ , where  $f \in (\mathfrak{t}_{\mathbf{R},2})_1$  satisfies  $\text{Alt}_2 f = \tilde{r}$ .<sup>1</sup>
- (iii) In  $\mathfrak{t}_{\mathbf{R},3}$ ,  $(\Phi_{\text{KZ}})_F = 1^{\otimes 3}$ , where

$$(\Phi_{\text{KZ}})_F = d_2^3(F) \cdot d_2^1(F) \cdot \Phi_{\text{KZ}} \cdot d_2^2(F)^{-1} d_2^0(F)^{-1}$$

- (iv)  $F$  satisfies

$$dF = \sum_{\alpha \in \mathbf{R}_+} \frac{d\alpha}{\alpha} \left( (\mathbf{K}_{\alpha,1} + \mathbf{K}_{\alpha,2}) \cdot F - F \cdot \mathbf{K}_{\alpha}^{(2)} \right) \quad (5.1)$$

**5.1.1. Compatibility with De Concini–Procesi associators.** Fix henceforth a positive, adapted family  $\beta = \{x_B\}_{B \subseteq D} \subset \mathfrak{h}^*$ . Let  $D$  be the Dynkin diagram corresponding to the root system  $\mathbf{R}$ . For any maximal nested set  $\mathcal{F} \in \text{Mns}(D)$ , let  $\Psi_{\mathcal{F}} : \mathcal{C} \rightarrow \mathfrak{t}_{\mathbf{R}}$  be the fundamental solution of  $\nabla_{\kappa}$  corresponding to  $\mathcal{F}$ , and  $\Upsilon_{\mathcal{G}\mathcal{F}} = \Psi_{\mathcal{G}}^{-1} \cdot \Psi_{\mathcal{F}}$  the corresponding De Concini–Procesi associator.

Let  $F_{\mathbf{R}} : \mathcal{C} \rightarrow \mathfrak{t}_{\mathbf{R},2}$  be a differential twist for  $\mathbf{R}$ , and set

$$F_{\mathcal{F}} = (\Psi_{\mathcal{F}}^{\otimes 2})^{-1} \cdot F \cdot d_1^1(\Psi_{\mathcal{F}})$$

The following is straightforward.

<sup>1</sup>There is an action of the symmetric group  $\mathfrak{S}_n$  on  $\mathfrak{t}_{\mathbf{R},n}$  defined by  $(k, k+1)\mathbf{K}_{\alpha}^{(n)} = \mathbf{K}_{\alpha}^{(n)}$  and

$$(k, k+1)\mathbf{K}_{\alpha,i} = \begin{cases} \mathbf{K}_{\alpha,i} & \text{if } i \neq k, k+1 \\ \mathbf{K}_{\alpha,i+1} & \text{if } i = k \\ \mathbf{K}_{\alpha,i-1} & \text{if } i = k+1 \end{cases}$$

$$(k, k+1)\Omega_{ij}^{\alpha} = \begin{cases} \Omega_{ij}^{\alpha} & \text{if } \{i, j\} \cap \{k, k+1\} = \emptyset \\ \Omega_{ij}^{-\alpha} & \text{if } \{i, j\} = \{k, k+1\} \\ \Omega_{i+1,j}^{\alpha} & \text{if } i = k \\ \Omega_{i-1,j}^{\alpha} & \text{if } i = k+1 \\ \Omega_{i,j+1}^{\alpha} & \text{if } j = k \\ \Omega_{i,j-1}^{\alpha} & \text{if } j = k+1 \end{cases}$$

**Lemma.** *The following holds*

- (i)  $F_{\mathcal{F}}$  is constant on  $\mathcal{C}$ .
- (ii)  $s_2^1(F_{\mathcal{F}}) = 1 = s_2^2(F_{\mathcal{F}})$ .
- (iii)  $(\Phi_{\text{KZ}})_{F_{\mathcal{F}}} = 1^{\otimes 3}$ .
- (iv)  $F_{\mathcal{F}} = \Phi_{\mathcal{F}\mathcal{G}}^{\otimes 2} \cdot F_{\mathcal{G}} \cdot d_1^1(\Phi_{\mathcal{G}\mathcal{F}})^{-1}$ .

5.1.2. *Notation.* Fix  $i \in \mathbf{I}$ , let  $\bar{\mathbf{R}} \subset \mathbf{R}$  be the root system generated by the simple roots  $\{\alpha_j\}_{j \neq i}$ ,  $\bar{\mathfrak{g}} \subset \mathfrak{g}$  the subalgebra spanned by the root vectors and coroots  $\{x_\alpha, \alpha^\vee\}_{\alpha \in \bar{\Phi}}$ ,  $\bar{\mathfrak{g}} \supset \bar{\mathfrak{h}}^e \subset \mathfrak{h}^e$  its essential Cartan subalgebra, and  $\bar{\mathfrak{l}} = \bar{\mathfrak{g}} \oplus \mathfrak{h}^e$  the corresponding Levi subalgebra of  $\mathfrak{g}$ . Similarly, we denote by  $\mathfrak{t}_{\bar{\mathbf{R}}} \subset \mathfrak{t}_{\mathbf{R}}$  the holonomy algebra of  $\bar{\mathbf{R}}$ .

The inclusion of root systems  $\bar{\mathbf{R}} \subset \mathbf{R}$  gives rise to a projection  $\pi : \mathfrak{h}^e \rightarrow \bar{\mathfrak{h}}^e$  determined by the requirement that  $\alpha(\pi(t)) = \alpha(t)$  for any  $\alpha \in \bar{\mathbf{R}}$ . The kernel of  $\pi$  is the line  $\mathbb{C}\lambda_i^\vee$  spanned by the  $i$ th fundamental coweight of  $\mathfrak{h}$ .

We shall coordinatise the fibres of  $\pi$  by restricting the simple root  $\alpha_i$  to them. This amounts to trivialising the fibration  $\pi : \mathfrak{h}^e \rightarrow \bar{\mathfrak{h}}^e$  as  $\mathfrak{h}^e \simeq \mathbb{C} \times \bar{\mathfrak{h}}^e$  via  $(\alpha_i, \pi)$ . The inverse of this isomorphism is given by  $(w, \bar{\mu}) \rightarrow w\lambda_i^\vee + \iota(\bar{\mu})$ , where  $\iota : \bar{\mathfrak{h}} \rightarrow \mathfrak{h}$  is the embedding with image  $\text{Ker}(\alpha_i)$  given by

$$\iota(\bar{t}) = \bar{t} - \alpha_i(\bar{t})\lambda_i^\vee \quad (5.2)$$

Denote by

$$\mathbf{K} = \sum_{\alpha \in \mathbf{R}_+} \mathbf{K}_\alpha \quad \text{and} \quad \bar{\mathbf{K}} = \sum_{\alpha \in \bar{\mathbf{R}}_+} \mathbf{K}_\alpha \quad (5.3)$$

the Casimir operators of  $\mathfrak{g}$  and  $\bar{\mathfrak{g}}$  in  $\mathfrak{t}_{\mathbf{R}}$  and  $\mathfrak{t}_{\bar{\mathbf{R}}}$ , respectively.

5.1.3. *Asymptotics of the Casimir connection for  $\alpha_i \rightarrow \infty$ .* Fix a vertex  $i \in D$ , and set  $\bar{D} = D \setminus \{\alpha_i\}$ . Fix  $\bar{\mu} \in \bar{\mathfrak{h}}^e = \mathfrak{h}_{\bar{D}}^e$ , and consider the fiber of  $\pi : \mathfrak{h}^e \rightarrow \bar{\mathfrak{h}}^e$  at  $\bar{\mu}$ . Since the restriction of  $\alpha \in \mathbf{R}$  to  $\pi^{-1}(\bar{\mu})$  is equal to  $\alpha(\lambda_i^\vee)\alpha_i + \alpha(\iota(\bar{\mu}))$ , the restriction of the Casimir connection  $\nabla$  to  $\pi^{-1}(\bar{\mu})$  is equal to

$$\nabla_{i, \bar{\mu}} = d - \sum_{\alpha \in \mathbf{R}_+ \setminus \bar{\mathbf{R}}} \frac{d\alpha_i}{\alpha_i - w_\alpha} \mathbf{K}_\alpha$$

where  $w_\alpha = -\alpha(\iota(\bar{\mu}))/\alpha(\lambda_i^\vee)$ . Set

$$R_{\bar{\mu}} = \max\{|w_\alpha|\}_{\alpha \in \mathbf{R} \setminus \bar{\mathbf{R}}} \quad (5.4)$$

**Proposition.**

- (i) *For any  $\bar{\mu} \in \bar{\mathfrak{h}}^e$ , there is a unique holomorphic function*

$$H_\infty : \{w \in \mathbb{P}^1 \mid |w| > R_{\bar{\mu}}\} \rightarrow \mathfrak{t}_{\mathbf{R}}$$

*such that  $H_\infty(\infty) = 1$  and, for any determination of  $\log(\alpha_i)$ , the function*

$$\Upsilon_\infty = H_\infty(\alpha_i) \cdot \alpha_i^{\mathbf{K} - \bar{\mathbf{K}}} \text{ satisfies}$$

$$\left( d - \sum_{\alpha \in \mathbf{R}_+ \setminus \bar{\mathbf{R}}} \frac{d\alpha_i}{\alpha_i - w_\alpha} \mathbf{K}_\alpha \right) \Upsilon_\infty = \Upsilon_\infty d$$

- (ii) The function  $H_\infty(\alpha_i, \bar{\mu})$  is holomorphic on the simply-connected domain  $\mathcal{D}_\infty \subset \mathbb{P}^1 \times \bar{\mathfrak{h}}$  given by

$$\mathcal{D}_\infty = \{(w, \bar{\mu}) \mid |w| > R_{\bar{\mu}}\} \quad (5.5)$$

and, as a function on  $\mathcal{D}_\infty$ ,  $\Upsilon_\infty$  satisfies

$$\left( d - \sum_{\alpha \in \mathbb{R}_+} \frac{d\alpha}{\alpha} \mathsf{K}_\alpha \right) \Upsilon_\infty = \Upsilon_\infty \left( d - \sum_{\alpha \in \bar{\mathbb{R}}_+} \frac{d\alpha}{\alpha} \mathsf{K}_\alpha \right)$$

5.1.4. *Asymptotics for  $\alpha_i \rightarrow \infty$  and fundamental De Concini–Procesi solutions.* Let  $\mathcal{F}$  be a maximal nested set on  $D$ , set  $\bar{\mathcal{F}} = \mathcal{F} \setminus \{D\}$  and  $\alpha_i = \alpha_{\mathcal{F}}^D$ . Let

$$\Psi_{\mathcal{F}} : \mathcal{C} \rightarrow \mathfrak{t}_{\mathbb{R}} \quad \text{and} \quad \Psi_{\bar{\mathcal{F}}} : \bar{\mathcal{C}} \rightarrow \mathfrak{t}_{\bar{\mathbb{R}}}$$

be the fundamental solutions of the Casimir connection for  $\mathbb{R}$  and  $\bar{\mathbb{R}} = \mathbb{R}_{D \setminus \alpha_i}$  corresponding to  $\mathcal{F}$ ,  $\bar{\mathcal{F}}$  respectively, and a positive, adapted family  $\{x_B\}_{B \subseteq D}$ . Regard  $\Psi_{\bar{\mathcal{F}}}$  as being defined on  $\mathcal{C}$  via the projection  $\pi : \mathfrak{h}^e \rightarrow \bar{\mathfrak{h}}^e$ . The result below expresses  $\Psi_{\mathcal{F}}$  in terms of  $\Psi_{\bar{\mathcal{F}}}$  and the solution  $\Upsilon_\infty$  given by Proposition 5.1.3.

**Proposition.** *The following holds*

$$\Psi_{\mathcal{F}} = \Upsilon_\infty \cdot \Psi_{\bar{\mathcal{F}}} \cdot x_D(\lambda_i^\vee)^{\mathsf{K} - \bar{\mathsf{K}}}$$

## 5.2. Differential twists and braided quasi–Coxeter structures.

5.2.1. *Relative twists.* Let  $F = F_{\mathbb{R}}$  be a differential twist for  $\mathbb{R}$ ,  $\alpha_i \in D$  a simple root, and  $\Upsilon_\infty$  the solution of the Casimir equations given by Proposition 5.1.3, where we are using the standard determination of log. Define  $F_\infty : \mathcal{C} \rightarrow \mathfrak{t}_{\mathbb{R},2}$  by

$$F_\infty = (\Upsilon_\infty^{\otimes 2})^{-1} \cdot F \cdot d_1^1(\Upsilon_\infty)$$

Then,  $F_\infty$  satisfies

- (i)  $s_2^1(F_\infty) = 1 = s_2^2(F_\infty)$ .  
(ii)

$$F_\infty = 1 + f \mod (\mathfrak{t}_{\mathbb{R},2})_{\geq 2}$$

where  $f \in (\mathfrak{t}_{\mathbb{R},2})_1$  satisfies  $\text{Alt}_2 f = \bar{r}$ .<sup>1</sup>

- (iii)  $(\Phi_{\mathsf{KZ}})_{F_\infty} = 1^{\otimes 3}$ .

- (iv)

$$dF_\infty = \sum_{\alpha \in \bar{\mathbb{R}}_+} \frac{d\alpha}{\alpha} \left( (\mathsf{K}_{\alpha,1} + \mathsf{K}_{\alpha,2}) \cdot F_\infty - F_\infty \cdot \mathsf{K}_\alpha^{(2)} \right)$$

Let  $\bar{\mathcal{C}}$  be the complexified chamber of  $\bar{\mathfrak{g}}$ , and  $\bar{F} = F_{\bar{\mathbb{R}}} : \bar{\mathcal{C}} \rightarrow \mathfrak{t}_{\bar{\mathbb{R}},2}$  a differential twist for  $\bar{\mathbb{R}}$ . Since the projection  $\pi : \mathfrak{h}^e \rightarrow \bar{\mathfrak{h}}^e$  maps  $\mathcal{C}$  to  $\bar{\mathcal{C}}$ , we may regard  $\bar{F}$  as a function on  $\mathcal{C}$ , and define  $F'_{(D;\alpha_i)} : \mathcal{C} \rightarrow \mathfrak{t}_{\mathbb{R},2}$  by

$$F'_{(D;\alpha_i)} = \bar{F}^{-1} \cdot F_\infty$$

**Proposition.**  *$F'_{(D;\alpha_i)}$  satisfies the following properties*

- (i)  $s_2^1(F'_{(D;\alpha_i)}) = 1 = s_2^2(F'_{(D;\alpha_i)})$ .

---

<sup>1</sup> $\bar{r} = \sum_{\alpha \in \bar{\mathbb{R}}_+} \Omega^\alpha - \Omega^{-\alpha}$ .

(ii)

$$F'_{(D;\alpha_i)} = 1 + f \mod (\mathfrak{t}_{\mathbb{R},2})_{\geq 2}$$

where  $f \in (\mathfrak{t}_{\mathbb{R},2})_1$  satisfies  $\text{Alt}_2 f = \tilde{r}_D - \tilde{r}_{D \setminus \alpha_i}$ <sup>1</sup>.

$$(iii) (\Phi_{\text{KZ}})_{F'_{(D;\alpha_i)}} = \Phi_{D \setminus \alpha_i}.$$

(iv)

$$dF'_{(D;\alpha_i)} = \sum_{\alpha \in \bar{\mathbb{R}}_+} \frac{d\alpha}{\alpha} [\mathbb{K}_\alpha^{(2)}, F'_{(D;\alpha_i)}]$$

In particular, if  $F'_{(D;\alpha_i)}$  is invariant under  $\mathfrak{t}_{\bar{\mathbb{R}}}$ , then it is constant on  $\mathcal{C}$ .

**5.2.2. Centraliser property.** Let  $\{F_B\}$  be a collection of differential twists for the subsystems  $\mathbb{R}_B \subset \mathbb{R}$ , where  $B$  is a subdiagram of  $D$ , such that if  $B$  has connected components  $\{B_i\}$ , then  $F_B = \prod_i F_{B_i}$ .

**Definition.** The collection  $\{F_B\}$  has the *centraliser property* if, for any  $\alpha \in B \subseteq D$ , the relative twist  $F_{(B,\alpha)}$  defined in 5.2.1 is invariant under  $\mathfrak{t}_{B \setminus \alpha}$  (and in particular constant).

**5.2.3. Factorisation.** Let  $\{F_B\}_{B \subseteq D}$  be a collection of differential twists with the centraliser property. For any  $\alpha_i \in B \subseteq D$ , set

$$F_{(B;\alpha_i)} = \left( x_B(\lambda_i^\vee)^{-(\mathbb{K}_B - \mathbb{K}_{B \setminus \alpha_i})} \right)^{\otimes 2} \cdot F'_{(B;\alpha_i)} \cdot d_1^1 \left( x_B(\lambda_{\alpha_i}^\vee)^{\mathbb{K}_B - \mathbb{K}_{B \setminus \alpha_i}} \right) \quad (5.6)$$

where  $F'_{(B;\alpha_i)} \in \mathfrak{t}_{B,2}$  is the relative twist defined in 5.2.1, and  $\{x_B\}_{B \subseteq D}$  is a positive, adapted family. The (constant) twist  $F_{(B;\alpha_i)}$  is invariant under  $\mathfrak{t}_{B \setminus \alpha_i}$ , and has the properties (1)–(4) given in Proposition 5.2.1. The following lemma is a direct consequence of Proposition 5.1.4.

**Lemma.** Let  $\mathcal{F}$  be a maximal nested set on  $D$ , and  $F_{\mathcal{F}}$  the twist defined in 5.1.1. Then, the following holds

$$F_{\mathcal{F}} = \prod_{B \in \mathcal{F}}^{\rightarrow} F_{(B;\alpha_{\mathcal{F}}^B)}$$

where the product is taken with  $F_{(B;\alpha_{\mathcal{F}}^B)}$  to the right of  $F_{(C;\alpha_{\mathcal{F}}^C)}$  if  $B \supset C$ .

**5.2.4. Weak braided quasi-Coxeter structure.** The following is a direct consequence of Proposition 5.2.1, and Lemma 5.2.3 and 5.1.1.

**Proposition.** Let  $\{F_B : \mathcal{C} \rightarrow \mathfrak{t}_{B,2}\}$  be a collection of differential twists satisfying the centraliser property. Then the elements  $\{F_{(B;\alpha)}\}$  defined in 5.2.3 satisfy

$$(i) s_2^1(F_{(B;\alpha_i)}) = 1 = s_2^2(F_{(B;\alpha_i)}).$$

(ii)

$$F_{(B;\alpha_i)} = 1 + f \mod (\mathfrak{t}_{\mathbb{R},2})_{\geq 2}$$

where  $f \in (\mathfrak{t}_{\mathbb{R},2})_1$  satisfies  $\text{Alt}_2 f = \tilde{r}_B - \tilde{r}_{B \setminus \alpha_i}$ .

$$(iii) (\Phi_{\text{KZ}})_{F_{(B;\alpha_i)}} = \Phi_{B \setminus \alpha_i}.$$

(iv) For any two  $\mathcal{F}, \mathcal{G} \in \text{Mns}(D)$ ,

$$F_{\mathcal{F}} = \Upsilon_{\mathcal{F}\mathcal{G}}^{\otimes 2} \cdot F_{\mathcal{G}} \cdot d_1^1(\Upsilon_{\mathcal{G}\mathcal{F}})$$

where  $F_{\mathcal{F}}, F_{\mathcal{G}}$  are defined as in 5.2.3.

In particular, the twists  $\{F_{(B;\alpha_i)}\}$  define a weak braided quasi-Coxeter structure on  $\mathfrak{t}_{\mathbb{R}}$ .

<sup>1</sup>For every  $B \subset D$ , we set  $\tilde{r}_B = \sum_{\alpha \in \mathbb{R}_B} \Omega^\alpha - \Omega^{-\alpha}$ .

### 5.3. The fusion operator.

5.3.1. *Dynamical KZ equation.* The dynamical KZ equation is the connection on the trivial bundle over  $\mathbb{C}^\times$  with fiber  $\mathfrak{t}_{\mathbb{R},2}$  given by

$$\nabla_z = d - \left( \frac{\Omega}{z} + \text{ad}\mu^{(1)} \right) dz \quad (5.7)$$

5.3.2. *The joint KZ–Casimir system.* The following connection on the trivial bundle over  $\mathbb{C}^\times \times \mathfrak{h}_{\text{reg}}^e$  with fiber  $\mathfrak{t}_{\mathbb{R},2}$  gives a non-trivial coupling of the KZ equations and the Casimir connection.

$$\nabla = d - \left( \frac{\Omega}{z} + \text{ad}\mu^{(1)} \right) dz - \sum_{\alpha \in \mathbb{R}_+} \mathsf{K}_\alpha^{(2)} \frac{d\alpha}{\alpha} - z \text{ad}\mu^{(1)}$$

The (exact) coupling term  $\text{ad}\mu^{(1)}dz + z \text{ad}\mu^{(1)} = d(z \text{ad}\mu^{(1)})$  was introduced in [11] when the equations take values in  $U\mathfrak{g}^{\otimes 2}$ , and the corresponding connection shown to be flat.

5.3.3. *Fundamental solution of the dynamical KZ equation near  $z = 0$ .* The following is straightforward.

**Proposition.**

- (i) For any  $\mu \in \mathfrak{h}$ , there is a unique holomorphic function  $H_0 : \mathbb{C} \rightarrow \mathfrak{t}_{\mathbb{R},2}$  such that  $H_0(0, \mu) \equiv 1$  and, for any determination of  $\log(z)$ , the  $\text{End}(\mathfrak{t}_{\mathbb{R},2})$ -valued function  $\Upsilon_0(z, \mu) = e^{z \text{ad}\mu^{(1)}} \cdot H_0(z, \mu) \cdot z^\Omega$  satisfies

$$\left( d_z - \left( \frac{\Omega}{z} + \text{ad}\mu^{(1)} \right) dz \right) \Upsilon_0 = \Upsilon_0 d_z$$

- (ii)  $H_0$  and  $\Upsilon_0$  are holomorphic functions in  $\mu$ , and  $\Upsilon_0$  satisfies

$$\left( d_{\mathfrak{h}} - \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \mathsf{K}_\alpha^{(2)} - z \text{ad}(d\mu^{(1)}) \right) \Upsilon_0 = \Upsilon_0 \left( d_{\mathfrak{h}} - \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \mathsf{K}_\alpha^{(2)} \right)$$

5.3.4. *Fundamental solutions of the dynamical KZ equations near  $z = \infty$ .* The connection (5.7) has an irregular singularity of Poincaré rank one at  $z = \infty$ . Let  $\mathbb{H}_\pm = \{z \in \mathbb{C} \mid \text{Im}(z) \gtrless 0\}$ . Let

$$\mathcal{C} = \{t \in \mathfrak{h}_{\mathbb{R}} \mid \alpha(t) > 0 \text{ for any } \alpha \in \mathbb{R}_+\}$$

be the fundamental Weyl chamber of  $\mathfrak{h}$ . Set  $\iota = \sqrt{-1}$ .

**Theorem** ([20]). For any  $\mu \in \mathcal{C}$ , there is a unique holomorphic function

$$H_\pm : \mathbb{H}_\pm \rightarrow \mathfrak{t}_{\mathbb{R},2}$$

such that  $H_\pm(z)$  tends to 1 as  $z \rightarrow \infty$  in any sector of the form  $|\arg(z)| \in (\delta, \pi - \delta)$ ,  $\delta > 0$  and, for any determination of  $\log(z)$ , the  $\text{End}(\mathfrak{t}_{\mathbb{R},2})$ -valued function

$$\Psi_\pm(z) = H_\pm(z) \cdot e^{z \text{ad}(\mu^{(1)})} \cdot z^{\Omega^0}$$

is a fundamental solution of the dynamical KZ connection  $\nabla_z$ .

5.3.5. *From the fusion operator to the differential twist.* Let  $\Psi_{\pm}$  be the fundamental solutions of  $\nabla_z$  given by Theorem 5.3.4, and define the *fusion operators*  $J_{\pm} : \mathbb{H}_{\pm} \times \mathcal{C} \rightarrow \mathfrak{t}_{R,2}$  by

$$J_{\pm}(z, \mu) = \Psi_{\pm}(z, \mu)(1)$$

**Theorem ([20]).** *Each of the holomorphic maps  $\overline{J}_{\pm} : \mathcal{C} \rightarrow \mathfrak{t}_{R,2}$  defined by*

$$\overline{J}_{\pm} = \Upsilon_0^{-1} \cdot J_{\pm}$$

*defines a differential twist for  $R$  satisfying the centraliser property. In particular, there is a weak quasi-Coxeter structure on  $\mathfrak{t}_R$  interpolating the monodromy of the KZ-Casimir system.*

## 6. QUANTUM GROUPS AND UNIVERSAL QUASI-COXETER STRUCTURES

In this section, we point out that the universal  $R$ -matrix of the Levi subalgebras of the quantum group  $U_{\hbar}\mathfrak{g}$  and its quantum Weyl group operators give rise to a braided quasi-Coxeter structure on the category  $\mathcal{O}_{\hbar}^{\text{int}}$  of integrable, highest weight modules. We then review the fact that this structure can be transferred to  $\mathcal{O}^{\text{int}}$  following [1]. The corresponding structure is universal, although in a different sense than that of Section 4. It arises from a PROP introduced in [2] which describes a bialgebra endowed with a root decomposition, together with a collection of Drinfeld–Yetter modules. The two different notions of universality will be related in Section 7.

6.1. **PROPS.** We briefly review the notion of *product-permutation category* (PROP), Drinfeld–Yetter modules of Lie bialgebras, and their propic description. For more details, we refer the reader to [8, 2]. Let  $k$  be a field of characteristic zero. A PROP  $(\mathcal{C}, S)$  is the datum of

- a strict, symmetric monoidal  $k$ -linear category  $\mathcal{C}$  whose objects are the non-negative integers, such that  $[n] \otimes [m] = [n + m]$ . In particular,  $[n] = [1]^{\otimes n}$  and  $[0]$  is the unit object;
- a bigraded set  $S = \bigsqcup_{m,n \in \mathbb{Z}_{\geq 0}} S_{nm}$  of morphism of  $\mathcal{C}$ , with

$$S_{nm} \subset \mathcal{C}([n], [m])$$

such that any morphism in  $\mathcal{C}$  can be obtained by composition, tensor product or linear combination over  $k$  of the morphisms in  $S$  and the permutation maps  $k\mathfrak{S}_n \subset \mathcal{C}([n], [n])$ .

Every PROP  $(\mathcal{C}, S)$  has a presentation in terms of generators and relations. Let  $\mathcal{F}_S$  be the PROP freely generated over  $S$ . There is a unique symmetric tensor functor  $\mathcal{F}_S \rightarrow \mathcal{C}$ , and  $\mathcal{C}$  has the form  $\mathcal{F}_S/\mathcal{I}$ , where  $\mathcal{I}$  is a tensor ideal in  $\mathcal{F}_S$  (i.e., a collection of subspaces  $\mathcal{I}_{nm} \subset \mathcal{C}([n], [m])$ , such that composition or tensor product (in any order) of any morphism in  $\mathcal{C}$  with any morphism in  $\mathcal{I}$  is still in  $\mathcal{I}$ ).

6.1.1. *Examples.*

- **Associative algebras.** Let  $\text{Alg}$  be the PROP  $\mathcal{F}_S/\mathcal{I}$ , where the set  $S$  consist of two elements  $\iota \in S_{0,1}$  (the unit) and  $m \in S_{2,1}$  (the multiplication), and  $\mathcal{I}$  is the ideal generated by the relations

$$\begin{aligned} m \circ (m \otimes \text{id}) &= m \circ (\text{id} \otimes m) \\ m \circ (\iota \otimes \text{id}) &= \text{id} = m \circ (\text{id} \otimes \iota) \end{aligned}$$

- **Lie algebras.** Let  $\mathbf{LA}$  be the PROP generated by the set  $S$  consisting of one element  $\mu \in S_{2,1}$  (the bracket) subject to the relations

$$\mu + \mu \circ (21) = 0 \quad \mu \circ (\mu \otimes \text{id}) \circ ((123) + (312) + (231)) = 0 \quad (6.1)$$

6.1.2. *The PROPs  $\underline{\mathbf{LCA}}$  and  $\underline{\mathbf{LBA}}$ .* The PROP of Lie coalgebras  $\underline{\mathbf{LCA}}$  is generated by a morphism  $\delta$  in bidegree  $(1, 2)$  with relations dual to (6.1), namely

$$\delta + (21) \circ \delta = 0 \quad \text{and} \quad ((123) + (312) + (231)) \circ (\delta \otimes \text{id}) \circ \delta = 0 \quad (6.2)$$

The PROP of Lie bialgebras  $\underline{\mathbf{LBA}}$  is generated by  $\mu$  in bidegree  $(2, 1)$  and  $\delta$  in bidegree  $(1, 2)$  satisfying (6.1), (6.2), and the *cocycle condition*

$$\delta \circ \mu = (\text{id} - (21)) \circ \text{id} \otimes \mu \circ \delta \otimes \text{id} \circ (\text{id} - (21)) \quad (6.3)$$

6.1.3. *Drinfeld doubles.* Let  $(\mathfrak{a}, [\cdot, \cdot]_{\mathfrak{a}}, \delta_{\mathfrak{a}})$  be a Lie bialgebra over  $\mathbf{k}$ . The Drinfeld double  $\mathfrak{g}_{\mathfrak{a}}$  of  $\mathfrak{a}$  is the Lie algebra defined as follows. As a vector space,  $\mathfrak{g}_{\mathfrak{a}} = \mathfrak{a} \oplus \mathfrak{a}^*$ . The pairing  $\langle \cdot, \cdot \rangle : \mathfrak{a} \otimes \mathfrak{a}^* \rightarrow \mathbf{k}$  extends uniquely to a symmetric, non-degenerate bilinear form on  $\mathfrak{g}_{\mathfrak{a}}$ , such that  $\mathfrak{a}, \mathfrak{a}^*$  are isotropic subspaces. The Lie bracket on  $\mathfrak{g}_{\mathfrak{a}}$  is then defined as the unique bracket compatible with  $\langle \cdot, \cdot \rangle$ , *i.e.*, such that

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle$$

for all  $x, y, z \in \mathfrak{g}_{\mathfrak{a}}$ . It coincides with  $[\cdot, \cdot]_{\mathfrak{a}}$  on  $\mathfrak{a}$ , and with the bracket induced by  $\delta_{\mathfrak{a}}$  on  $\mathfrak{a}^*$ . The mixed bracket for  $a \in \mathfrak{a}, b \in \mathfrak{a}^*$  is equal to

$$[a, b] = \text{ad}^*(b)(a) - \text{ad}^*(a)(b)$$

where  $\text{ad}^*$  denotes the coadjoint action of  $\mathfrak{a}^*$  on  $\mathfrak{a}$  and of  $\mathfrak{a}$  on  $\mathfrak{a}^*$ , respectively.

The Lie algebra  $\mathfrak{g}_{\mathfrak{a}}$  is a (topological) quasitriangular Lie bialgebra, with cobracket  $\delta = \delta_{\mathfrak{a}} \oplus (-\delta_{\mathfrak{a}^*})$ , where  $\delta_{\mathfrak{a}^*}$  is the (topological) cobracket on  $\mathfrak{a}^*$  induced by  $[\cdot, \cdot]_{\mathfrak{a}}$ , and  $r$ -matrix  $r \in \mathfrak{g}_{\mathfrak{a}} \hat{\otimes} \mathfrak{g}_{\mathfrak{a}}$  corresponding to the identity in  $\text{End}(\mathfrak{a}) \simeq \mathfrak{a} \hat{\otimes} \mathfrak{a}^* \subset \mathfrak{g}_{\mathfrak{a}} \hat{\otimes} \mathfrak{g}_{\mathfrak{a}}$ . Explicitly, if  $\{a_i\}_{i \in I}, \{b^i\}_{i \in I}$  are dual bases of  $\mathfrak{a}$  and  $\mathfrak{a}^*$  respectively, then  $r = \sum_{i \in I} a_i \otimes b^i \in \mathfrak{a} \hat{\otimes} \mathfrak{a}^*$ .

6.1.4. *Drinfeld–Yetter modules.* A Lie bialgebra  $(\mathfrak{a}, [\cdot, \cdot]_{\mathfrak{a}}, \delta_{\mathfrak{a}})$  has a natural category of representations called Drinfeld–Yetter modules, and denoted  $\mathbf{DY}_{\mathfrak{a}}$ . A triple  $(V, \pi, \pi^*)$  is a Drinfeld–Yetter module on  $\mathfrak{a}$  if  $(V, \pi)$  is an  $\mathfrak{a}$ -module, that is the map  $\pi : \mathfrak{a} \otimes V \rightarrow V$  satisfies

$$\pi \circ \mu = \pi \circ (\text{id} \otimes \pi) - \pi \circ (\text{id} \otimes \pi) \circ (21) \quad (6.4)$$

$(V, \pi^*)$  is an  $\mathfrak{a}$ -comodule, that is the map  $\pi^* : V \rightarrow \mathfrak{a} \otimes V$  satisfies

$$\delta \circ \pi^* = (21) \circ (\text{id} \otimes \pi^*) \circ \pi^* - (\text{id} \otimes \pi^*) \circ \pi^* \quad (6.5)$$

and the maps  $\pi, \pi^*$  satisfy the following compatibility condition in  $\text{End}(\mathfrak{a} \otimes V)$ :

$$\pi^* \circ \pi - \text{id} \otimes \pi \circ (12) \circ \text{id} \otimes \pi^* = [\cdot, \cdot]_{\mathfrak{a}} \otimes \text{id} \circ \text{id} \otimes \pi^* - \text{id} \otimes \pi \circ \delta_{\mathfrak{a}} \otimes \text{id} \quad (6.6)$$

The category  $\mathbf{DY}_{\mathfrak{a}}$  is a symmetric tensor category, and is equivalent to the category  $\mathcal{E}_{\mathfrak{g}_{\mathfrak{a}}}$  of equicontinuous  $\mathfrak{g}_{\mathfrak{a}}$ -modules [7]. A  $\mathfrak{g}_{\mathfrak{a}}$ -module is equicontinuous if the action of the  $b^i$ 's is locally finite, *i.e.*, for every  $v \in V$ ,  $b^i \cdot v = 0$  for all but finitely many  $i \in I$ . In particular, given  $(V, \pi) \in \mathcal{E}_{\mathfrak{g}_{\mathfrak{a}}}$ , the coaction of  $\mathfrak{a}$  on  $V$  is given by

$$\pi^*(v) = \sum_i a_i \otimes b^i \cdot v \in \mathfrak{a} \otimes V$$



where the equicontinuity condition ensures that the sum is finite. The action of the  $r$ -matrix of  $\mathfrak{g}_a$  on the tensor product  $V \otimes W \in \mathcal{E}_{\mathfrak{g}_a}$  corresponds, under the identification  $\mathcal{E}_{\mathfrak{g}_a} \simeq \mathrm{DY}_a$  with the map  $r_{VW} : V \otimes W \rightarrow V \otimes W$  given by

$$r_{VW} = \pi_V \otimes \mathrm{id} \circ (12) \circ \mathrm{id} \otimes \pi_W^* \quad (6.7)$$

**6.2. The Casimir algebra.** We now review the construction of the *universal Casimir algebra*  $\mathfrak{U}_R$  introduced in [2]. For some generalities on PROPs and Lie bialgebras, we refer the reader to the Appendix 6.1.

**6.2.1. The Casimir category.** Let  $R$  be a fixed root system of rank  $l$  with simple roots  $\{\alpha_1, \dots, \alpha_l\}$ , and  $R_+ \subset R$  the corresponding positive roots.

Let  $\widetilde{\mathrm{LBA}}_R$  be the PROP with generators  $\mu \in S_{21}, \delta \in S_{12}, \pi_\alpha \in S_{11}, \alpha \in R_+$ . The morphism  $\mu$  and  $\delta$  satisfy the relations

$$\mu + \mu \circ (21) = 0 \quad \mu \circ (\mu \otimes \mathrm{id}) \circ ((123) + (312) + (231)) = 0 \quad (6.8)$$

$$\delta + (21) \circ \delta = 0 \quad ((123) + (312) + (231)) \circ (\delta \otimes \mathrm{id}) \circ \delta = 0 \quad (6.9)$$

and are related by the cocycle condition

$$\delta \circ \mu = (\mathrm{id} - (21)) \circ \mathrm{id} \otimes \mu \circ \delta \otimes \mathrm{id} \circ (\mathrm{id} - (21)) \quad (6.10)$$

The morphisms  $\{\pi_\alpha\}_{\alpha \in R_+}$  satisfy

$$\pi_\alpha \circ \pi_\beta = \delta_{\alpha, \beta} \pi_\alpha \quad (6.11)$$

The compatibility condition with  $\mu$  and  $\delta$  is given by the relation

$$\pi_\alpha \circ \mu = \sum_{\beta + \gamma = \alpha} \mu \circ (\pi_\beta \otimes \pi_\gamma) \quad (6.12)$$

and

$$\delta \circ \pi_\alpha = \sum_{\beta + \gamma = \alpha} (\pi_\beta \otimes \pi_\gamma) \circ \delta \quad (6.13)$$

We wish to impose the additional completeness relation

$$\sum_{\alpha \in R_+} \pi_\alpha = \mathrm{id}_{[1]} \quad (6.14)$$

To this end, let  $p \in \mathbb{N}$ , and denote by  $\mathbf{k}[\mathbf{R}_+^p]^{\mathrm{fin}}$  the algebra of functions on  $\mathbf{R}_+^p$  with finite support. The vector space  $\widetilde{\mathrm{LBA}}_R([p], [q])$  is naturally a  $(\mathbf{k}[\mathbf{R}_+^q]^{\mathrm{fin}}, \mathbf{k}[\mathbf{R}_+^p]^{\mathrm{fin}})$ -bimodule. More specifically, let  $\underline{\alpha} = (\alpha_{i_1}, \dots, \alpha_{i_p}) \in \mathbf{R}_+^p$  and  $\delta_{\underline{\alpha}} \in \mathbf{k}[\mathbf{R}_+^p]$  the characteristic function on  $\underline{\alpha}$ . Then  $\delta_{\underline{\alpha}}$  acts on  $\widetilde{\mathrm{LBA}}_R([p], [q])$  by precomposition with the map

$$\pi_{\underline{\alpha}} = \bigotimes_{k=1}^p \pi_{\alpha_{i_k}} : [p] \rightarrow [p]$$

Similarly for  $\mathbf{k}[\mathbf{R}_+^q]$ .

Let  $\mathrm{LBA}_R$  be the (topological) PROP with morphisms

$$\mathrm{LBA}_R([p], [q]) = \mathbf{k}[\mathbf{R}_+^p] \otimes_{\mathbf{k}[\mathbf{R}_+^p]^{\mathrm{fin}}} \widetilde{\mathrm{LBA}}_R([p], [q]) \otimes_{\mathbf{k}[\mathbf{R}_+^q]^{\mathrm{fin}}} \mathbf{k}[\mathbf{R}_+^q] \quad (6.15)$$

Then, the function  $\mathbf{1} \in \mathbf{k}[\mathbf{R}_+^p]$  acts on  $[p]$  as  $\mathrm{id}_{[p]}$ .

**Definition.** The Casimir category  $\underline{\mathrm{LBA}}_R$  is the Karoubi envelope of the (topological) PROP  $\mathrm{LBA}_R$ .

**EXAMPLE.** Let  $\mathfrak{g}$  be Kac–Moody algebra of with root system  $R$ . Then the positive Borel subalgebra  $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha$  is an  $\underline{\mathbf{LBA}}_R$ -module.

### 6.2.2. The Casimir algebra.

**Definition.** The category  $\mathbf{DY}_R^n$ ,  $n \geq 0$ , is the multicolored PROP generated by  $n + 1$  objects  $[1]$  and  $\underline{V}_k$ ,  $k = 1, \dots, n$ , and morphisms

$$\begin{aligned} \mu : [2] &\rightarrow [1] & \delta : [1] &\rightarrow [2] & \pi_\alpha : [1] &\rightarrow [1], \alpha \in R_+ \\ \pi_k : [1] \otimes \underline{V}_k &\rightarrow \underline{V}_k & \pi_k^* : \underline{V}_k &\rightarrow [1] \otimes \underline{V}_k \end{aligned}$$

such that  $([1], \mu, \delta, \pi_\alpha)$  is a  $\underline{\mathbf{LBA}}_R$ -module in  $\mathbf{DY}_R^n$ , and, for every  $k = 1, \dots, n$ ,  $(\underline{V}_k, \pi_k, \pi_k^*)$  is a Drinfeld–Yetter module over  $[1]$ .

**Definition.** The  $n$ -Casimir algebra is the algebra of endomorphisms

$$\mathfrak{U}_R^n = \text{End}_{\mathbf{DY}_R^n} \left( \bigotimes_{k=1}^n \underline{V}_k \right)$$

**6.2.3. Casimir algebra and completions.** Let  $\mathfrak{g}$  be a Kac–Moody algebra with root system  $R$ ,  $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha$  the corresponding positive Borel subalgebra, and  $\mathfrak{g}_\mathfrak{b} = \mathfrak{g} \oplus \mathfrak{h}$  the Drinfeld double of  $\mathfrak{b}$ . Let  $\mathbf{DY}_\mathfrak{b}$  be the category of Drinfeld–Yetter modules over  $\mathfrak{b}$ . For any  $n$ -tuple  $\{V_k, \pi_k, \pi_k^*\}_{k=1}^n$  of Drinfeld–Yetter  $\mathfrak{b}$ -modules, there is a unique tensor functor

$$\mathcal{G}_{(\mathfrak{b}, V_1, \dots, V_n)} : \mathbf{DY}_R^n \longrightarrow \mathbf{Vect}_k$$

such that  $[1] \mapsto \mathfrak{b}, \underline{V}_k \mapsto V_k$ .

**Proposition.** Let  $f : \mathbf{DY}_\mathfrak{b} \rightarrow \mathbf{Vect}_k$  be the forgetful functor, and  $\mathcal{U}_\mathfrak{b}^n = \text{End}(f^{\boxtimes n})$ . The functors  $\mathcal{G}_{(\mathfrak{b}, V_1, \dots, V_n)}$  induce an algebra homomorphism

$$\rho_\mathfrak{b}^n : \mathfrak{U}_R^n \rightarrow \mathcal{U}_\mathfrak{b}^n$$

**6.2.4. Cosimplicial structure.** The tower of algebras  $\{\mathcal{U}_\mathfrak{b}^n\}$  is naturally endowed with a cosimplicial structure, as described in 4.2.1. This can be lifted to the algebras  $\mathfrak{U}_R^n$ .

For every  $n \geq 1$  and  $i = 0, 1, \dots, n + 1$ , there are faithful functors

$$\mathcal{D}_n^{(i)} : \underline{\mathbf{DY}}_n \rightarrow \underline{\mathbf{DY}}_{n+1}$$

given by

$$\mathcal{D}_n^{(i)} = \begin{cases} \mathcal{G}_{([1], \underline{V}_2, \dots, \underline{V}_{n+1})} & i = 0 \\ \mathcal{G}_{([1], \underline{V}_1, \dots, \underline{V}_i \otimes \underline{V}_{i+1}, \dots, \underline{V}_{n+1})} & 1 \leq i \leq n \\ \mathcal{G}_{([1], \underline{V}_1, \dots, \underline{V}_n)} & i = n + 1 \end{cases}$$

These induce algebra homomorphisms

$$\Delta_n^{(i)} : \mathfrak{U}_R^n \rightarrow \mathfrak{U}_R^{n+1}$$

which are universal analogues of the insertion/coproduct maps on  $U\mathfrak{g}_\mathfrak{b}^{\otimes n}$ . They give the tower of algebras  $\{\mathfrak{U}_R^n\}_{n \geq 0}$  a cosimplicial structure. The morphisms  $\rho_\mathfrak{b}^n : \mathfrak{U}_R^n \rightarrow \mathcal{U}_\mathfrak{b}^n$  defined in 6.2.3 are compatible with the face morphisms.

6.2.5. *A basis of  $\mathfrak{U}_R^n$ .* For any  $p \in \mathbb{N}$  and  $\underline{p} = (p_1, \dots, p_n) \in \mathbb{N}^n$  such that  $|\underline{p}| = p_1 + \dots + p_n = p$ , define the maps

$$\pi^{(\underline{p})} : [p] \otimes \bigotimes_{k=1}^n \underline{\mathbb{V}}_k \rightarrow \bigotimes_{k=1}^n \underline{\mathbb{V}}_k \quad (6.16)$$

as the ordered composition of  $p_i$  actions on  $\underline{\mathbb{V}}_i$ . Similarly for

$$\pi^{*(\underline{p})} : \bigotimes_{k=1}^n \underline{\mathbb{V}}_k \rightarrow [p] \otimes \bigotimes_{k=1}^n \underline{\mathbb{V}}_k \quad (6.17)$$

The following provides an explicit basis of the algebra  $\mathfrak{U}_R^n = \text{End}_{\text{DY}_R^n}(\bigotimes \underline{\mathbb{V}}_k)$ .

**Proposition.** *The set of endomorphisms of  $\underline{\mathbb{V}}_1 \otimes \dots \otimes \underline{\mathbb{V}}_n$*

$$r_{\underline{N}, \underline{N}'}^{\underline{\alpha}, \sigma} = \pi^{(\underline{N})} \circ \pi_{\underline{\alpha}} \otimes \text{id} \circ \sigma \otimes \text{id} \circ \pi^{*(\underline{N}')}$$

for  $N \geq 0$ ,  $\sigma \in \mathfrak{S}_N$ ,  $\underline{\alpha} \in \mathbb{R}_+^N$ ,  $\underline{N}, \underline{N}' \in \mathbb{N}^n$  such that  $|\underline{N}| = N = |\underline{N}'|$ , is a basis of  $\mathfrak{U}_R^n$ .

6.2.6. *A diagrammatic representation of the morphisms in  $\text{DY}_R^n$ ,  $n = 1$ .* The endomorphisms of  $\underline{\mathbb{V}}_1 \in \text{DY}_R^1$  given by

$$r_{N, N}^{\underline{\alpha}, \sigma} = \pi^{(N)} \circ \pi_{\underline{\alpha}} \otimes \text{id} \circ \sigma \otimes \text{id} \circ \pi^{*(N)}$$

for  $N \geq 0$  and  $\sigma \in \mathfrak{S}_N$  are a basis of  $\mathfrak{U}_R = \text{End}_{\text{DY}_R^1}(\underline{\mathbb{V}}_1)$ .

We represent the morphisms  $\mu, \delta, \pi, \pi^*$  in  $\underline{\text{DY}}_1$  with the oriented diagrams

$$\mu : \begin{array}{c} \nearrow \quad \searrow \\ \downarrow \quad \downarrow \\ \rightarrow \end{array} \quad \delta : \begin{array}{c} \rightarrow \quad \rightarrow \\ \nearrow \quad \searrow \\ \downarrow \end{array}$$

and

$$\pi : \begin{array}{c} \nearrow \\ \downarrow \\ \text{---} \end{array} \quad \pi^* : \begin{array}{c} \text{---} \\ \downarrow \\ \searrow \end{array}$$

The morphisms  $\pi_{\alpha}$  are represented by

$$\pi_{\alpha} : \text{---} \bigcirc \alpha \text{---}$$

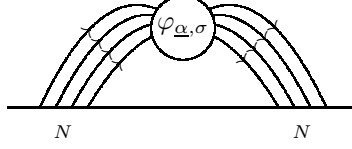
A non-trivial endomorphism of  $\underline{\mathbb{V}}_1$  is represented as a linear combination of oriented diagrams, necessarily starting with a coaction and ending with an action. The compatibility relation (6.6)

$$\begin{array}{c} \nearrow \quad \searrow \\ \downarrow \quad \downarrow \end{array} = \begin{array}{c} \nearrow \quad \searrow \\ \searrow \quad \nearrow \\ \downarrow \end{array} + \begin{array}{c} \nearrow \quad \searrow \\ \nearrow \quad \searrow \\ \downarrow \end{array} - \begin{array}{c} \nearrow \quad \searrow \\ \nearrow \quad \searrow \\ \downarrow \end{array}$$

allows to reorder  $\pi$  and  $\pi^*$ . The cocycle condition (6.3) allows to reorder brackets and cobrackets as in LBA. Finally, the relations (6.4), (6.5)

$$\begin{array}{c} \nearrow \quad \searrow \\ \downarrow \quad \downarrow \end{array} = \begin{array}{c} \nearrow \quad \searrow \\ \searrow \quad \nearrow \\ \downarrow \end{array} - \begin{array}{c} \nearrow \quad \searrow \\ \searrow \quad \nearrow \\ \downarrow \end{array} \\ \begin{array}{c} \nearrow \quad \searrow \\ \nearrow \quad \searrow \\ \downarrow \end{array} = \begin{array}{c} \nearrow \quad \searrow \\ \searrow \quad \nearrow \\ \downarrow \end{array} - \begin{array}{c} \nearrow \quad \searrow \\ \searrow \quad \nearrow \\ \downarrow \end{array}$$

allow to remove from the graph every  $\mu$  and every  $\delta$  involved. It follows that every endomorphism of  $\underline{V}_1$  is a linear combination of the elements  $r_{N,N}^{\underline{\alpha},\sigma}$



where  $\varphi_{\underline{\alpha},\sigma} = \sigma \circ \pi_{\underline{\alpha}}$ , for some  $N \geq 0$ ,  $\sigma \in \mathfrak{S}_N$ , and  $\underline{\alpha} \in R_+^N$ .

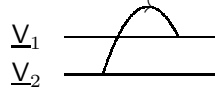
**6.2.7. The  $r$ -matrix and the Casimir element.** The equivalence between the category of Drinfeld–Yetter  $\mathfrak{g}$ -modules and the category of equicontinuous  $\mathfrak{g}_{\mathfrak{b}}$ -modules gives an isomorphism between the algebra  $\mathcal{U}_{\mathfrak{b}}^n$  and a completion of the universal enveloping algebra  $U\mathfrak{g}_{\mathfrak{b}}^{\otimes n}$ . In particular, under this identification, the action of the classical  $r$ -matrix of  $\mathfrak{g}_{\mathfrak{b}}$  on a tensor product  $V \otimes W$  of equicontinuous  $\mathfrak{g}_{\mathfrak{b}}$ -modules is identified with the endomorphism

$$\pi_V \otimes \text{id} \circ \sigma_{12} \circ \text{id} \otimes \pi_W^*$$

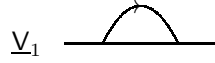
and the action of the normally ordered Casimir on  $V$  is identified with the endomorphism

$$\pi_V \circ \pi_V^*$$

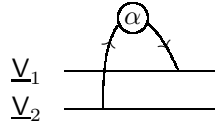
These special elements belongs in fact to the image of  $\rho_{\mathfrak{b}}^n : \mathcal{U}_{\mathfrak{R}}^n \rightarrow \mathcal{U}_{\mathfrak{b}}^n$ . In particular, the  $r$ -matrix is the image of the element of  $\mathcal{U}_{\mathfrak{R}}^2$



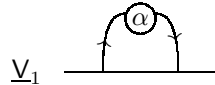
and the normally ordered Casimir is the image of the element of  $\mathcal{U}_{\mathfrak{R}}$



Moreover the elements  $\Omega^{\alpha} \in \mathcal{U}_{\mathfrak{b}}^2$ ,  $\alpha \in R \cup \{0\}$  correspond to the diagrams



and the elements  $\mathcal{K}_{\alpha}^+ \in \mathcal{U}_{\mathfrak{b}}$ ,  $\alpha \in R_+$  correspond to the diagrams



**6.2.8.  $D$ -algebra structure.** Let  $D$  be the Dynkin diagram associated to the root system  $R$ . We associate to any subdiagram  $B \subset D$  the root subsystem  $R_B \subset R$ , and the corresponding subalgebra  $\mathcal{U}_B = \mathcal{U}_{R_B} \subset \mathcal{U}_R$ .

**Proposition.** *The collection of subalgebras  $\{\mathcal{U}_B\}_{B \subset D}$  defines a  $D$ -algebra structure on  $\mathcal{U}_R$ .*

6.2.9. *Weak quasi-Coxeter structures on  $\mathfrak{U}_R$ .*

**Definition.** A *weak Coxeter structure* on  $\mathfrak{U}_R$  is the datum of

- (i) for each connected subdiagram  $B \subseteq D$ , an  $R$ -matrices  $R_B \in \mathfrak{U}_B^2$  and associator  $\Phi_B \in \mathfrak{U}_B^3$  of the form

$$R_B = e^{\Omega_B/2} \quad \text{and} \quad \Phi_B = \Phi'_B(\Omega_{B,12}, \Omega_{B,23})$$

where  $\Phi'_B$  is a Lie associator.

- (ii) for each pair of subdiagrams  $B' \subseteq B \subseteq D$  and maximal nested set  $\mathcal{F} \in \text{Mns}(B', B)$ , a relative twist  $J_{\mathcal{F}}^{B B'} \in (\mathfrak{U}_B^2)^{B'}$ , satisfying

$$J_{\mathcal{F}}^{B B'} = 1 \mod (\mathfrak{U}_B^2)_{\geq 1}^{B'} \quad (6.18)$$

and

$$(\Phi_B)_{J_{\mathcal{F}}^{B B'}} = \Phi_{B'} \quad (6.19)$$

which is compatible with *vertical decomposition*

$$J_{\mathcal{F}_1 \cup \mathcal{F}_2}^{B B''} = J_{\mathcal{F}_1}^{B B'} \cdot J_{\mathcal{F}_2}^{B' B''}$$

where  $B'' \subseteq B' \subseteq B$ ,  $\mathcal{F}_1 \in \text{Mns}(B, B')$  and  $\mathcal{F}_2 \in \text{Mns}(B', B'')$ .

- (iii) for any  $B' \subseteq B$  and pair of maximal nested sets  $\mathcal{F}, \mathcal{G} \in \text{Mns}(B, B')$ , a gauge transformation, referred to as a *De Concini–Procesi associators*,  $\Upsilon_{\mathcal{F}\mathcal{G}} \in \mathfrak{U}_B^{B'}$ , satisfying

$$\Upsilon_{\mathcal{F}\mathcal{G}} = 1 \mod (\mathfrak{U}_B^{B'})_{\geq 1} \quad (6.20)$$

$$\Upsilon_{\mathcal{F}\mathcal{G}} \otimes \Upsilon_{\mathcal{F}\mathcal{G}} \cdot J_{\mathcal{G}} \cdot \Delta(\Upsilon_{\mathcal{F}\mathcal{G}})^{-1} = J_{\mathcal{F}} \quad (6.21)$$

and such that the following holds

- **Orientation:** for any  $\mathcal{F}, \mathcal{G} \in \text{Mns}(B, B')$

$$\Upsilon_{\mathcal{F}\mathcal{G}} = \Upsilon_{\mathcal{G}\mathcal{F}}^{-1}$$

- **Transitivity:** for any  $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \text{Mns}(B, B')$

$$\Upsilon_{\mathcal{H}\mathcal{F}} = \Upsilon_{\mathcal{H}\mathcal{G}} \Upsilon_{\mathcal{G}\mathcal{F}}$$

- **Factorisation:**

$$\Upsilon_{(\mathcal{F}_1 \cup \mathcal{F}_2)(\mathcal{G}_1 \cup \mathcal{G}_2)} = \Upsilon_{\mathcal{F}_1 \mathcal{G}_1} \Upsilon_{\mathcal{F}_2 \mathcal{G}_2}$$

for any  $B'' \subseteq B' \subseteq B$ ,  $\mathcal{F}_1, \mathcal{G}_1 \in \text{Mns}(B, B')$  and  $\mathcal{F}_2, \mathcal{G}_2 \in \text{Mns}(B', B'')$ .

**6.3. Universal Coxeter structures on Kac–Moody algebra.** Let  $\mathfrak{b}$  be the positive Borel subalgebra of a Kac–Moody algebra with root system  $R$ . Let  $\text{DY}_{\mathfrak{b}}^{\text{int}}$  be the category of deformation integrable  $\mathfrak{h}$ –diagonalisable Drinfeld–Yetter  $\mathfrak{b}$ –modules, and let  $\mathfrak{U}$  be the algebra of endomorphisms of the fiber functor from  $\text{DY}_{\mathfrak{b}}^{\text{int}}$  to the category of  $\mathbb{C}[[\hbar]]$ –modules.

We pointed out in [2] that a weak quasi-Coxeter structure on  $\mathfrak{U}_R$  induces a weak quasi-Coxeter structure on  $\mathcal{U}_{\mathfrak{b}}$  (called *universal*) and therefore on the category  $\text{DY}_{\mathfrak{b}}^{\text{int}}$ , through the morphism of cosimplicial algebras  $\mathfrak{U}_R^n \rightarrow \mathcal{U}_{\mathfrak{b}}^n$ .

Let  $\mathfrak{b}_i$ ,  $i \in D$ , be the Borel subalgebras corresponding to the simple roots  $\alpha_i$ ,  $i \in D$ , and  $\mathcal{U}_i$  the corresponding completed cosimplicial algebras.

**Definition.** A *universal Coxeter structure* on  $\mathrm{DY}_{\mathfrak{b}}^{\mathrm{int}}$  is the data of a universal weak Coxeter structure on  $\mathcal{U}_{\mathfrak{b}}$  and a collection of operators, called *local monodromies*,  $S_i \in \mathcal{U}_i$ ,  $i \in D$ , of the form

$$S_i = \tilde{s}_i \cdot \underline{S}_i \quad (6.22)$$

where  $\underline{S}_i \in \mathcal{U}_i^{\mathfrak{h}}$ ,  $\underline{S}_i = 1 \pmod{\hbar}$ , and  $\tilde{s}_i = \exp(e_i) \exp(-f_i) \exp(e_i)$ , satisfying the coproduct identity

$$\Delta_{J_i}(S_i) = (R_i)_{J_i}^{21}(S_i \otimes S_i) \quad (6.23)$$

and the generalized braid relations of type  $D$ . Namely, for any pairs  $i, j$  of distinct vertices of  $B$ , such that  $2 < m_{ij} < \infty$ , and elementary pair  $(\mathcal{F}, \mathcal{G})$  in  $\mathrm{Mns}(B)$  such that  $i \in \mathcal{F}, j \in \mathcal{G}$ , the following relations hold in  $\mathcal{U}_{\mathfrak{b}}$ ,

$$\mathrm{Ad}(\Upsilon_{\mathcal{G}\mathcal{F}})(S_i) \cdot S_j \cdots = S_j \cdot \mathrm{Ad}(\Upsilon_{\mathcal{G}\mathcal{F}})(S_i) \cdots \quad (6.24)$$

where the number of factors in each side equals  $m_{ij}$ .

**6.4. The standard quasi-Coxeter structure on  $U_{\hbar}\mathfrak{g}$ .** Let  $\mathfrak{g}$  be a symmetrisable Kac-Moody algebra with root system  $R$ , Dynkin diagram  $D$  and positive Borel subalgebra  $\mathfrak{b}$ . Then, the collection of diagrammatic subalgebras  $\{\mathfrak{g}_B\}_{B \subset D}$  and  $\{\mathfrak{b}_B\}_{B \subset D}$  endow, respectively,  $\mathfrak{g}$  and  $\mathfrak{b}$  with a standard  $D$ -algebra structure. Let  $U_{\hbar}\mathfrak{b}_B$  be the Drinfeld-Jimbo quantum group corresponding to  $\mathfrak{b}_B$ , and  $\mathrm{DY}_{U_{\hbar}\mathfrak{b}_B}^{\mathrm{int}}$  the category of  $\mathfrak{h}$ -diagonalizable integrable Drinfeld-Yetter  $U_{\hbar}\mathfrak{b}_B$ -modules.

The universal  $R$ -matrix and the quantum Weyl group operators induce on  $\mathrm{DY}_{\mathfrak{b}}^{\mathrm{int}}$  a *standard* braided quasi-Coxeter structure. More specifically,

- (i) the universal  $R$ -matrix  $R_B^{\hbar}$  of  $U_{\hbar}\mathfrak{g}_B$  endows  $\mathrm{DY}_{U_{\hbar}\mathfrak{b}_B}^{\mathrm{int}}$  with a braided tensor structure;
- (ii) the restriction functors  $\mathrm{Res}_B^{B'} : \mathrm{DY}_{U_{\hbar}\mathfrak{b}_B}^{\mathrm{int}} \rightarrow \mathrm{DY}_{U_{\hbar}\mathfrak{b}_{B'}}^{\mathrm{int}}$  endow the collection of braided tensor categories  $\mathrm{DY}_{U_{\hbar}\mathfrak{b}_B}^{\mathrm{int}}$  with a structure of braided  $D$ -category;
- (iii) finally, the quantum Weyl group operators  $S_i^{\hbar} \in \mathrm{DY}_{U_{\hbar}\mathfrak{b}_i}^{\mathrm{int}}$  defined in [15] satisfy the braid relations and the coproduct identity

$$\Delta_{\hbar}(S_i^{\hbar}) = R_i^{21} \cdot (S_i^{\hbar} \otimes S_i^{\hbar})$$

and endow  $\mathrm{DY}_{U_{\hbar}\mathfrak{b}}^{\mathrm{int}}$  with a structure of (strict) braided quasi-Coxeter category.

In [7, 8, 9], Etingof and Kazhdan constructed an equivalence of braiding tensor categories

$$\mathrm{EK} : \mathrm{DY}_{\mathfrak{b}} \xrightarrow{\sim} \mathrm{DY}_{U_{\hbar}\mathfrak{b}}$$

where  $\mathrm{DY}_{\mathfrak{b}}$  is the category of deformation Drinfeld-Yetter  $\mathfrak{b}$ -modules. The functor  $\mathrm{EK}$  preserves the weight decomposition, and it provides an equivalence at the level of category  $\mathcal{O}$

$$\mathrm{EK} : \mathcal{O} \xrightarrow{\sim} \mathcal{O}_{\hbar}$$

In [1], we generalised their construction and we used the resulting equivalence to transfer the braided quasi-Coxeter structure of  $U_{\hbar}\mathfrak{b}$  to the category  $\mathrm{DY}_{\mathfrak{b}}^{\mathrm{int}}$ .

**Theorem.** [1]

- (i) *There exists a structure of braided quasi-Coxeter category on  $\mathrm{DY}_{\mathfrak{b}}^{\mathrm{int}}$  and an equivalence of quasi-Coxeter categories*

$$\Psi : \mathrm{DY}_{\mathfrak{b}}^{\mathrm{int}} \xrightarrow{\sim} \mathrm{DY}_{U_{\hbar}\mathfrak{b}}^{\mathrm{int}}$$

*whose underlying tensor functors are  $\mathrm{EK}_B$ .*

- (ii) The equivalence  $\Psi$  descends to an equivalence of braided quasi-Coxeter categories

$$\Psi : \mathcal{O}^{\text{int}} \xrightarrow{\sim} \mathcal{O}_h^{\text{int}}$$

- (iii) The braided quasi-Coxeter structure induced on  $\text{DY}_{\mathfrak{b}}^{\text{int}}$  and  $\mathcal{O}^{\text{int}}$  by the quantum group  $U_h \mathfrak{g}$  is universal, i.e., it comes from a weak quasi-Coxeter structure on  $\mathfrak{U}_{\mathbf{R}}$ .

## 7. THE MONODROMY THEOREM

This section contains the main result of this paper. We prove that the double holonomy algebras  $\mathfrak{t}_{\mathbf{R},n}$ , which underlie the braided quasi-Coxeter structure on the category  $\mathcal{O}^{\text{int}}$ , maps to the Casimir algebras  $\mathfrak{U}_{\mathbf{R}}^n$ , which underlie the braided quasi-Coxeter structure transferred from the quantum group  $U_h \mathfrak{g}$ . We then use the rigidity statement of [2] to prove that these two structures are equivalent, and in particular that the monodromy of the Casimir connection of  $\mathfrak{g}$  is given by the quantum Weyl group operators of  $U_h \mathfrak{g}$ .

### 7.1. The double holonomy algebra and the Casimir algebra.

7.1.1. *Commutation statements in  $\mathfrak{U}_{\mathbf{R}}$ .* Set  $\kappa = \pi \circ \pi^*$  and  $\kappa_{\alpha} = \pi \circ \pi_{\alpha} \otimes \text{id} \circ \pi^*$ ,  $\alpha \in \mathbf{R}_+ \cup \{0\}$ . The following is proved in [2].

**Proposition.**

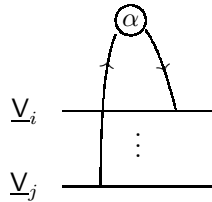
- (i)  $\kappa$  is central in  $\mathfrak{U}_{\mathbf{R}}$ .
- (ii) For any  $\alpha \in \mathbf{R}_+$ ,  $[\kappa_0, \kappa_{\alpha}] = 0$ .
- (iii) For any  $B \subseteq D$ ,  $\alpha \in \mathbf{R}_B$ ,

$$[\kappa_{\alpha}, \sum_{\beta \in \mathbf{R}_B} \kappa_{\beta}] = 0$$

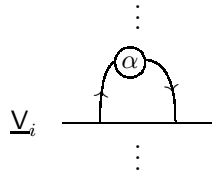
- (iv) For any subsystem  $\Psi \subset \mathbf{R}_+$ ,  $\text{rk}(\Psi) = 2$ ,  $\alpha \in \Psi$ ,

$$[\kappa_{\alpha}, \sum_{\beta \in \Psi} \kappa_{\beta}] = 0$$

7.1.2. *From the double holonomy algebra to the Casimir algebra.* Let  $\Omega_{ij}^{\alpha}$ ,  $\alpha \in \mathbf{R}_+ \cup 0$  be the endomorphism of  $\underline{\mathbf{V}}_1 \otimes \cdots \otimes \underline{\mathbf{V}}_n$  in  $\mathfrak{U}_{\mathbf{R}}^2$



and set  $\Omega_{ij}^{-\alpha} = \Omega_{ji}^{\alpha}$ . Let  $\kappa_{\alpha,i} \in \mathfrak{U}_{\mathbf{R}}^n$  be the endomorphism of  $\underline{\mathbf{V}}_1 \otimes \cdots \otimes \underline{\mathbf{V}}_n$



**Proposition.**

- (i) The linear maps  $\eta_{R,n} : \mathfrak{t}_{R,n} \rightarrow \mathfrak{U}_R^n$  defined by

$$\eta_{R,n}(\Omega_{ij}^\alpha) = \Omega_{ij}^\alpha \quad \eta_{R,n}(K_\alpha^{(n)}) = \Delta^{(n)}(\kappa_\alpha) \quad \eta_{R,n}(K_{\alpha,i}) = \kappa_{\alpha,i}$$

are morphisms of algebras, compatible with the cosimplicial structure and the natural gradation on  $\mathfrak{t}_{R,n}, \mathfrak{U}_R^n$ .

- (ii) The following

$$\begin{array}{ccc} \mathfrak{t}_{R,n} & \xrightarrow{\xi_{R,n}} & \mathcal{U}_b^n \\ \eta_{R,n} \downarrow & \nearrow \rho_b^n & \\ \mathfrak{U}_R^n & & \end{array}$$

is a commutative diagram of morphisms of cosimplicial algebras.

- (iii)

*Proof.* (i) The result follows directly from the commutation relation of 7.1.1.

(ii) The commutativity follows by direct inspection.  $\square$

**7.2. Equivalence of quasi-Coxeter structures.** We now prove the main result of the paper. In the previous section we constructed a weak quasi-Coxeter structure on  $\mathfrak{U}_R$ , and therefore on  $\mathfrak{U}$ , interpolating the monodromy of the KZ and the Casimir connection.

**Theorem.**

- (i) The weak braided quasi-Coxeter structure on  $\mathfrak{U}_R$  coming from the quantum group  $U_{\hbar}\mathfrak{g}$  is equivalent to the one constructed in Sections 3, 4 and 5, induced by the KZ and the Casimir connections.  $\nabla_{KZ}, \nabla_C$ .
- (ii) The differential weak braided quasi-Coxeter structure on  $\mathfrak{U}_R$  extends uniquely to a braided quasi-Coxeter structure on  $\mathcal{U}_b$  with local monodromies

$$S_{i,C} = \tilde{s}_i e^{\frac{\hbar}{2} C_i} \quad i \in D$$

- (iii) There exists an equivalence of quasi-Coxeter categories

$$\Psi : \mathrm{DY}_{\mathfrak{b}}^{\nabla_{KZ}, \nabla_C} \rightarrow \mathrm{DY}_{U_{\hbar}\mathfrak{b}}^{R_{\hbar}, S_{\hbar}}$$

where  $\mathrm{DY}_{U_{\hbar}\mathfrak{b}}^{R_{\hbar}, S_{\hbar}}$  is the category  $\mathrm{DY}_{U_{\hbar}\mathfrak{b}}^{\mathrm{int}}$  endowed with the standard quasi-Coxeter structure induced by the quantum group  $U_{\hbar}\mathfrak{g}$ , and  $\mathrm{DY}_{\mathfrak{b}}^{\nabla_{KZ}, \nabla_C}$  is the category  $\mathrm{DY}_{\mathfrak{b}}^{\mathrm{int}}$  endowed with the differential quasi-Coxeter structure induced by  $\nabla_{KZ}$  and  $\nabla_C$  and extended as in (ii).

- (iv) The braid group representation defined by the action of the quantum Weyl group operators is equivalent to the monodromy representation of the Casimir connection defined in Section 2.

*Proof.* (i) The result follows by 7.1.2 and by uniqueness of the weak quasi-Coxeter structure on the Casimir algebra. Namely, we proved in [2] the following

- (a) Up to a unique equivalence, there exists a unique weak braided quasi-Coxeter structure on  $\mathfrak{U}_R$ .
- (b) A weak braided quasi-Coxeter structure on  $\mathfrak{U}_R$  can be completed to at most one braided quasi-Coxeter structure on  $\mathcal{U}_b$ .

(ii) The operators  $S_{i,C}$  satisfy the coproduct identity (6.23). By (b), they satisfy the braid relations (6.24) and extend the weak structure to a braided quasi-Coxeter structure on  $\mathcal{U}_b$ .



- (iii) The equivalence follows from Theorem 6.4, (ii), and by (b).  
 (iv) Clear.  $\square$

#### APPENDIX A. THE CASIMIR CONNECTION OF AN AFFINE KAC-MOODY ALGEBRA

In this section, we prove that every affine Lie algebra admits two equivariant extensions of the Casimir connection,  $A_\kappa := A + A_{\mathfrak{h}}$  and  $A_C := A_\kappa + A_{S^2\mathfrak{h}}$ , where

$$A = \frac{\hbar}{2} \sum_{\alpha \in \mathbb{R}_+} \mathcal{K}_\alpha^+ \frac{d\alpha}{\alpha}$$

such that

- (i) the connections,  $\nabla = d - A_\kappa$  and  $\nabla = d - A_C$  are flat and  $W$ -equivariant;
- (ii)  $\text{Res}_{\alpha_i=0} A_\kappa = \kappa_i/2$ ,  $i \in I$ ;
- (ii)  $\text{Res}_{\alpha_i=0} A_C = C_i/2$ ,  $i \in I$ ;

where  $\kappa_i$  and  $C_i$  are, respectively, the truncated and the full Casimir element of  $\mathfrak{sl}_2^{\alpha_i}$ .

**A.1. The Dilogarithm.** For any  $\delta \in \mathbb{C}^\times$ , set

$$\Psi_\delta^\pm(x) = \sum_{n>0} \left( \frac{1}{\pm x + n\delta} - \frac{1}{n\delta} \right) = \Psi_\delta^\mp(-x)$$

**Lemma.**

- (i)  $\Psi_\delta^\pm(x)$  is holomorphic on  $\mathbb{C} \setminus \mathbb{Z}_{\neq 0}\delta$
- (ii)  $\Psi_\delta^+(x + \delta) = \Psi_\delta^+(x) - \frac{1}{x + \delta}$
- (iii)  $\Psi_\delta^-(x + \delta) = \Psi_\delta^-(x) - \frac{1}{x}$

Set  $\Psi^\pm = \Psi_1^\pm$  and  $\Psi = \Psi^+ + \Psi^-$ .

**A.2. The form  $A_{\mathfrak{h}}$ .** Let  $\mathfrak{g}$  be an affine Kac-Moody of rank  $l + 1$  with root system  $\mathbb{R}$ , and let  $\mathfrak{g}$  be the corresponding finite dimensional Lie algebra with root system  $\mathbb{R}$ . Let  $\mathfrak{h} \subset \mathfrak{g}$  be the unique realization of the associated Cartan matrix, and

$$\mathfrak{h} = \mathring{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

where  $\mathring{\mathfrak{h}} \subset \mathfrak{g}$ ,  $c = \sum_{i=0}^l a_i^\vee \alpha_i^\vee$  and  $d$  is defined by  $\alpha_i(d) = \delta_{i,0}$ . Let  $(-, -)$  be the normalized non-degenerate bilinear form on  $\mathfrak{h}$ , and  $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$  the isomorphism induced by  $(-, -)$  [12]. Let  $\delta = \sum_{i=0}^l a_i \alpha_i$ . Set

$$A_{\mathfrak{h}} = \sum_{\beta \in \mathring{\mathbb{R}}_+} A_\beta \left( \frac{\beta}{\delta} \right) + B \frac{d\delta}{\delta} \quad (\text{A.1})$$

where

$$A_\beta \left( \frac{\beta}{\delta} \right) = \frac{1}{2} \left[ \left( \frac{\delta}{\beta} + \Psi \left( \frac{\beta}{\delta} \right) \right) h_\beta - \frac{\beta}{\delta} \left( 2 + \Psi \left( \frac{\beta}{\delta} \right) \right) c \right] d \left( \frac{\beta}{\delta} \right) \quad (\text{A.2})$$

and  $B = \rho^\vee \in \mathfrak{h}$  is a fixed solution of

$$\langle B, \alpha_i \rangle = 1 \quad i = 0, 1, \dots, l \quad (\text{A.3})$$

**Theorem.** *The form  $A + A_{\mathfrak{h}}$  defines a flat and  $W$ -equivariant connection with residues*

$$\text{Res}_{\alpha_i=0} A + A_{\mathfrak{h}} = f_i e_i + \frac{1}{2} h_i = \frac{1}{2} \kappa_i$$

### A.3. Proof of Theorem A.2.

A.3.1. Let first consider the case  $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ . We first assume that  $A_{\mathfrak{h}}$  has the form

$$A_{\mathfrak{h}} = \left( S \left( \frac{\theta}{\delta} \right) h + T \left( \frac{\theta}{\delta} \right) c \right) d \left( \frac{\theta}{\delta} \right) + B(\delta) d\delta$$

Then  $A_{\mathfrak{h}}$  is closed and

$$A_{\mathfrak{h}} = \frac{1}{\delta} \left( S \left( \frac{\theta}{\delta} \right) h + T \left( \frac{\theta}{\delta} \right) c \right) d\theta - \frac{\theta}{\delta} \left( S \left( \frac{\theta}{\delta} \right) h + T \left( \frac{\theta}{\delta} \right) c \right) d\delta + B(\delta) d\delta$$

Let  $W^{\text{ext}}$  the *extended* Weyl group, i.e.,  $W^{\text{ext}} = W \rtimes \text{Aut}(D_{\mathfrak{g}})$ .

The  $W^{\text{ext}}$ -equivariance of  $A + A_{\mathfrak{h}}$  is equivalent to the conditions

$$s_1^* A_{\mathfrak{h}} = A_{\mathfrak{h}} - \frac{h}{\theta} d\theta \quad (\text{A.4})$$

$$\gamma^* A_{\mathfrak{h}} = A_{\mathfrak{h}} \quad (\text{A.5})$$

where  $s_1$  is the simple reflection on  $\theta$  and  $\gamma$  is induced by the symmetry of the Dynkin diagram of  $\widehat{\mathfrak{sl}_2}$ . In particular,

$$\begin{aligned} s_1(\theta) &= -\theta & s_1(\delta) &= \delta & s_1(\Lambda) &= \Lambda \\ \gamma(\theta) &= -\theta + \delta & \gamma(\delta) &= \delta & \gamma(\Lambda) &= \frac{\theta}{2} - \frac{c}{4} + \Lambda \end{aligned} \quad (\text{A.6})$$

A.3.2. The conditions (A.4), (A.5) are equivalent to the system of equations ( $z = \theta/\delta$ )

$$S(-z) = S(z) - \frac{1}{z} \quad (\text{A.7})$$

$$-T(-z) = T(z) \quad (\text{A.8})$$

$$S(1-z) = S(z) \quad (\text{A.9})$$

$$T(z) + T(1-z) = -S(1-z) \quad (\text{A.10})$$

and

$$\frac{z}{\delta} S(-z) + (s_1^* B(\delta))_{(h)} = -\frac{z}{\delta} S(z) + B(\delta)_{(h)} \quad (\text{A.11})$$

$$\frac{z}{\delta} T(-z) + (s_1^* B(\delta))_{(c)} = -\frac{z}{\delta} T(z) + B(\delta)_{(c)} \quad (\text{A.12})$$

$$-\frac{z}{\delta} S(1-z) + (\gamma^* B(\delta))_{(h)} = -\frac{z}{\delta} S(z) + B(\delta)_{(h)} \quad (\text{A.13})$$

$$\frac{z}{\delta} [S(1-z) + T(1-z)] + (\gamma^* B(\delta))_{(c)} = -\frac{z}{\delta} T(z) + B(\delta)_{(c)} \quad (\text{A.14})$$

A.3.3. Assume the existence of functions  $S, T$  satisfying (A.7), (A.8), (A.9), (A.10). Then the function  $B(\delta)$  should satisfy

$$s_1^* B(\delta) = B(\delta) - \frac{h}{\delta} \quad (\text{A.15})$$

$$\gamma^* B(\delta) = B(\delta) \quad (\text{A.16})$$

The general solution is easily computed to be

$$B(\delta) = \frac{1}{\delta} \left( \frac{h}{2} + 2d + f(\delta)c \right)$$

where  $f(\delta)$  is any function in  $\delta$ . In particular, it satisfies the condition (A.3).

A.3.4. Assume now the existence of a function  $S$  satisfying (A.7), (A.9). Then if we force  $T$  to be of the form

$$T(z) = p(z)S(z) + q(z)$$

where  $p, q$  are two polynomials in  $z$ , then (A.8), (A.10) are equivalent to the system

$$\begin{aligned} p(z) + p(-z) &= 0 \\ q(z) + q(-z) &= \frac{1}{z}p(-z) \\ p(z) + p(1-z) &= -1 \\ q(z) + q(1-z) &= 0 \end{aligned}$$

A solution is given by

$$p(z) = -z \quad \text{and} \quad q(z) = \frac{1}{2} - z$$

It follows that the general solution for  $T$  has the form

$$T(z) = -z(S(z) + 1) + \frac{1}{2} + E(z)$$

where  $E(z)$  is any function satisfying

$$\begin{aligned} E(-z) &= -E(z) \\ E(z) &= -E(1-z) \end{aligned}$$

A.3.5. Finally, we need to solve the equations (A.7) and (A.9), which are equivalent to the system

$$\begin{aligned} S(-z) &= S(z) - \frac{1}{z} \\ S(z+1) &= S(z) - \frac{1}{z} \end{aligned}$$

A particular solution is given by

$$S(z) = \frac{1}{2} \left( \frac{1}{z} + \Psi(z) \right)$$

It follows that the general solution to (A.7) and (A.9) is given by the formula

$$S(z) = \frac{1}{2} \left( \frac{1}{z} + \Psi(z) \right) + e(z)$$

where  $e(z)$  is any function satisfying

$$\begin{aligned} e(-z) &= e(z) \\ e(z+1) &= e(z) \end{aligned}$$

A.3.6. Setting  $e = E = f = 0$ , we get, for  $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ ,

$$A_{\mathfrak{h}} = A_{\theta} = \frac{1}{2} \left[ \left( \frac{\delta}{\theta} + \Psi \left( \frac{\theta}{\delta} \right) \right) h - \frac{\theta}{\delta} \left( 2 + \Psi \left( \frac{\theta}{\delta} \right) \right) c \right] d \left( \frac{\theta}{\delta} \right) + \left( \frac{h}{2} + 2d \right) \frac{d\delta}{\delta}$$

and the resulting connection  $\nabla = d - (A + A_{\mathfrak{h}})$  is flat and  $W$ -equivariant. A simple computation shows that

$$\begin{aligned} \text{Res}_{\theta=0} A + A_{\mathfrak{h}} &= \frac{1}{2} \kappa_{\theta} d(\theta) \\ \text{Res}_{\theta=\delta} A + A_{\mathfrak{h}} &= \frac{1}{2} \kappa_{\delta-\theta} d(\delta - \theta) \end{aligned}$$

A.3.7. Let now  $\mathfrak{g}$  be an affine Kac–Moody algebra and set

$$A_{\mathfrak{h}} = \sum_{\beta \in \check{\mathbf{R}}_+} A_{\beta} \left( \frac{\beta}{\delta} \right) + B \frac{d\delta}{\delta}$$

where

$$A_{\beta} \left( \frac{\beta}{\delta} \right) = \frac{1}{2} \left[ \left( \frac{\delta}{\beta} + \Psi \left( \frac{\beta}{\delta} \right) \right) h_{\beta} - \frac{\beta}{\delta} \left( 2 + \Psi \left( \frac{\beta}{\delta} \right) \right) c \right] d \left( \frac{\beta}{\delta} \right)$$

and  $B \in \mathfrak{h}$ .

A.4. It follows from the case  $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$  that the form  $A_{\beta}(\beta/\delta)$  satisfies

$$A_{-\beta} \left( \frac{-\beta}{\delta} \right) = A_{\beta} \left( \frac{\beta}{\delta} \right) - \frac{h_{\beta}}{\beta} d\beta + \frac{h_{\beta}}{\delta} d\delta \quad (\text{A.17})$$

$$A_{-\beta+\delta} \left( \frac{-\beta+\delta}{\delta} \right) = A_{\beta} \left( \frac{\beta}{\delta} \right) \quad (\text{A.18})$$

A.5. For every  $i = 1, \dots, l$ , the simple reflection  $s_i$  permutes the elements in  $\check{\mathbf{R}}_+ \setminus \{\alpha_i\}$ , and

$$s_i^* \left( \sum_{\beta \in \check{\mathbf{R}}_+} A_{\beta} \left( \frac{\beta}{\delta} \right) \right) = \sum_{\beta \in \check{\mathbf{R}}_+ \setminus \{\alpha_i\}} A_{\beta} \left( \frac{\beta}{\delta} \right) + A_{-\alpha_i} \left( \frac{-\alpha_i}{\delta} \right)$$

By (A.17),

$$s_i^* \left( \sum_{\beta \in \check{\mathbf{R}}_+} A_{\beta} \left( \frac{\beta}{\delta} \right) \right) = \sum_{\beta \in \check{\mathbf{R}}_+} A_{\beta} \left( \frac{\beta}{\delta} \right) - \frac{h_i}{\alpha_i} d\alpha_i + \frac{h_i}{\delta} d\delta$$

Therefore the  $\check{W}$ -equivariance of the form  $A + A_{\mathfrak{h}}$  is equivalent to the condition

$$s_i(B) = B - h_i \iff \langle \alpha_i, B \rangle = 1 \quad (\text{A.19})$$

A.6. Let  $\beta \in \check{\mathbf{R}}_+$ . Then we say that  $\beta \in R_k$ ,  $k = 0, -1, -2$ , if  $\langle \beta, h_0 \rangle = k$ . In particular,  $R_{-2} = \{\theta\}$ , and

$$s_0(\beta) = \begin{cases} \beta & \text{if } k = 0 \\ -(\theta - \beta) + \delta & \text{if } k = -1 \\ -\theta + 2\delta & \text{if } k = -2 \end{cases}$$

It follows from (A.18) that

$$A_{-(\theta-\beta)+\delta} \left( \frac{-(\theta-\beta)+\delta}{\delta} \right) = A_{\theta-\beta} \left( \frac{\theta-\beta}{\delta} \right)$$

Therefore

$$s_0^* \left( \sum_{\beta \in \tilde{\mathbf{R}}_+} A_\beta \left( \frac{\beta}{\delta} \right) \right) = \sum_{\beta \in \tilde{\mathbf{R}}_+ \setminus \{\theta\}} A_\beta \left( \frac{\beta}{\delta} \right) + A_{-\theta+2\delta} \left( \frac{-\theta+2\delta}{\delta} \right)$$

By (A.17) and (A.18),

$$\begin{aligned} A_{-\theta+2\delta} \left( \frac{-\theta+2\delta}{\delta} \right) &= A_{\theta-\delta} \left( \frac{\theta-\delta}{\delta} \right) = \\ &= A_{\delta-\theta} \left( \frac{\delta-\theta}{\delta} \right) - \frac{h_0}{\alpha_0} d\alpha_0 + \frac{h_0}{\delta} d\delta = \\ &= A_\theta \left( \frac{\theta}{\delta} \right) - \frac{h_0}{\alpha_0} d\alpha_0 + \frac{h_0}{\delta} d\delta \end{aligned}$$

The condition  $s_0^*(A + A_{\mathfrak{h}}) = A + A_{\mathfrak{h}}$  is therefore equivalent to

$$s_0(B) = B - h_0 \iff \langle B, \alpha_0 \rangle = 1$$

A.7. It follows that for any choice of a solution  $B = \rho^\vee \in \mathfrak{h}$  of the equations

$$\langle B, \alpha_i \rangle = 1 \quad i = 0, 1, \dots, l$$

the form  $A_\kappa = A + A_{\mathfrak{h}}$ , where

$$A_{\mathfrak{h}} = \sum_{\beta \in \tilde{\mathbf{R}}_+} A_\beta \left( \frac{\beta}{\delta} \right) + B \frac{d\delta}{\delta} \quad (\text{A.20})$$

and

$$A_\beta \left( \frac{\beta}{\delta} \right) = \frac{1}{2} \left[ \left( \frac{\delta}{\beta} + \Psi \left( \frac{\beta}{\delta} \right) \right) h_\beta - \frac{\beta}{\delta} \left( 2 + \Psi \left( \frac{\beta}{\delta} \right) \right) c \right] d \left( \frac{\beta}{\delta} \right) \quad (\text{A.21})$$

defines a flat,  $W$ -equivariant connection with residues

$$\text{Res}_{\alpha_i=0} A_\kappa = \frac{1}{2} \kappa_i d\alpha_i$$

The theorem is proved.

A.8. **The form  $A_{S^2\mathfrak{h}}$ .** We now show that the equivariant connection  $\nabla = d - (A + A_{\mathfrak{h}})$  can be extended with a closed,  $W$ -equivariant one form  $A_{S^2\mathfrak{h}}$  with values in  $S^2\mathfrak{h}$ , in order to correct the residues and obtain

$$\text{Res}_{\alpha_i=0} A + A_{\mathfrak{h}} + A_{S^2\mathfrak{h}} = \frac{1}{2} C_i d\alpha_i$$

This provides an analogue of the (finite-dimensional) Casimir connection with coefficients  $C_\alpha$ .

Set

$$A_{S^2\mathfrak{h}} = \sum_{\beta \in \tilde{\mathbf{R}}_+} \frac{\pi}{2} \cot \left( \pi \frac{\beta}{\delta} \right) (h_\beta - \frac{\beta}{\delta} c)^2 d \left( \frac{\beta}{\delta} \right) \quad (\text{A.22})$$

**Theorem.** *The form  $A_{S^2\mathfrak{h}}$  is closed and  $W$ -equivariant. The form  $A_C = A + A_{\mathfrak{h}} + A_{S^2\mathfrak{h}}$  defines a flat and  $W$ -equivariant connection with residues*

$$\text{Res}_{\alpha_i=0} A_C = \frac{1}{2} \kappa_i + \frac{1}{2} h_{\alpha_i} = \frac{1}{2} C_i$$

*Proof.* Let first consider the case  $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ . We have

$$A_{S^2\mathfrak{h}} = \frac{\pi}{2} \cot\left(\pi \frac{\theta}{\delta}\right) \left(h - \frac{\theta}{\delta}c\right)^2 d\left(\frac{\theta}{\delta}\right)$$

$A_{S^2\mathfrak{h}}$  is closed with residues

$$\text{Res}_{\theta=0} A_{S^2\mathfrak{h}} = \frac{1}{2} h^2 d\theta \quad (\text{A.23})$$

$$\text{Res}_{\theta=\delta} A_{S^2\mathfrak{h}} = \frac{1}{2} (-h+c)^2 d(\delta-\theta) \quad (\text{A.24})$$

It remains to prove that the form is  $W$ -equivariant. It is enough to observe that

$$s_1^* A_{S^2\mathfrak{h}} = \frac{\pi}{2} \cot\left(\pi \frac{-\theta}{\delta}\right) \left(-h - \frac{-\theta}{\delta}c\right)^2 d\left(\frac{-\theta}{\delta}\right) = A_{S^2\mathfrak{h}}$$

and

$$s_0^* A_{S^2\mathfrak{h}} = \frac{\pi}{2} \cot\left(\pi \frac{-\theta+2\delta}{\delta}\right) \left(-h+2c - \frac{-\theta+2\delta}{\delta}c\right)^2 d\left(\frac{-\theta+2\delta}{\delta}\right) = A_{S^2\mathfrak{h}}$$

Let now  $\mathfrak{g}$  be an affine Kac–Moody algebra. Then we set

$$A_{S^2\mathfrak{h}} = \sum_{\beta \in \mathbb{R}_+} \frac{\pi}{2} \cot\left(\pi \frac{\beta}{\delta}\right) \left(h_\beta - \frac{\beta}{\delta}c\right)^2 d\left(\frac{\beta}{\delta}\right) \quad (\text{A.25})$$

Clearly,  $A_{S^2\mathfrak{h}}$  is closed with the required residues. In order to prove the  $W$ -equivariance it is enough to observe that

$$\sum_{\beta \in \mathbb{R}_+} \frac{\pi}{2} \cot\left(\pi \frac{\beta}{\delta}\right) \left(h_\beta - \frac{\beta}{\delta}c\right)^2 d\left(\frac{\beta}{\delta}\right) = \frac{\delta}{4} \sum_{\alpha \in \mathbb{R}_+^{\text{re}}} \left(\frac{1}{\alpha} - \frac{1}{w(\alpha)}\right) \left(h_\alpha - \frac{\alpha}{\delta}c\right)^2 d\left(\frac{\alpha}{\delta}\right)$$

for any element of the Weyl group  $w \in W$ . The result follows. <sup>1</sup>  $\square$

**Remark.** The expression of the form  $A_{S^2\mathfrak{h}}$  for  $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$  has been computed in the following way. We first assume

$$A_{S^2\mathfrak{h}} = A_\theta(\theta, \delta) d\theta + A_\delta(\theta, \delta) d\delta \quad (\text{A.26})$$

where

$$A_\theta(\theta, \delta) = S(\theta, \delta) h^2 + T(\theta, \delta) h c + U(\theta, \delta) c^2$$

and similarly for  $A_\delta$ . The condition of  $W$ -equivariance (for a fixed value  $\delta \in \mathbb{C}^*$ ) gives a system of difference equation in  $\theta$  for the functions  $S, T, U$  which is easily solved with functions of the form  $p(x) \cot(x)$ , where  $p$  is a polynomial. More

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<sup>1</sup>In the case of  $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ , one has

$$\begin{aligned} \pi \cot\left(\pi \frac{\theta}{\delta}\right) d\left(\frac{\theta}{\delta}\right) &= \delta \left[ \frac{1}{\theta} + \sum_{n>0} \left( \frac{1}{\theta+n\delta} - \frac{1}{-\theta+n\delta} \right) \right] d\left(\frac{\theta}{\delta}\right) = \\ &= \frac{\delta}{2} \left( \frac{1}{\theta} - \frac{1}{s_1(\theta)} \right) d\left(\frac{\theta}{\delta}\right) + \frac{\delta}{2} \sum_{n>0} \left( \frac{1}{\theta+n\delta} - \frac{1}{s_1(\theta+n\delta)} \right) d\left(\frac{\theta+n\delta}{\delta}\right) \\ &\quad + \frac{\delta}{2} \sum_{n>0} \left( \frac{1}{s_1(-\theta+n\delta)} - \frac{1}{-\theta+n\delta} \right) d\left(-\frac{-\theta+n\delta}{\delta}\right) = \\ &= \frac{\delta}{2} \sum_{\alpha \in \mathbb{R}_+^{\text{re}}} \left( \frac{1}{\alpha} - \frac{1}{s_1(\alpha)} \right) d\left(\frac{\alpha}{\delta}\right) \end{aligned}$$

Similarly for higher rank  $\mathfrak{g}$  and  $w \in W$ .

specifically,  $S, U$  are odd functions in  $\theta$ ,  $T$  is an even function in  $\theta$  (invariance with respect to  $s_1$ ) such that

$$S(\theta + \delta) = S(\theta) \quad (\text{A.27})$$

$$T(\theta + \delta) = T(\theta) - 2S(\theta) \quad (\text{A.28})$$

$$U(\theta + \delta) = U(\theta) + S(\theta) - T(\theta) \quad (\text{A.29})$$

where the system follows from the invariance with respect to the translation  $\theta \mapsto \theta - \delta$ .

The condition  $dA = 0$  then provides a formula for  $A_\delta$ . Specifically, one gets to a general solution of the form

$$A_{S^2\mathfrak{h}} = \frac{\pi}{2} \cot\left(\pi \frac{\theta}{\delta}\right) \left(h - \frac{\theta}{\delta}c\right)^2 d\left(\frac{\theta}{\delta}\right) + B(\delta)d\delta \quad (\text{A.30})$$

where  $B(\delta)$  is a  $W$ -equivariant function (which has been chosen equal to zero).

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